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Solving Nonlinear Time Delay Control Systems by Fourier series

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Abstract: *In this paper we present a method to find the solution of time-delay optimal control systems using Fourier series. The method is based upon expanding various time functions in the system as their truncated Fourier series. Operational matrices of integration and delay are presented and are utilized to reduce the solution of time-delay control systems to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.*
Keywords: Fourier series, Time delay system, Operational matrix, Nonlinear systems.

I. Introduction

The control of systems with time delay has been of considerable concern. Delays occur frequently in biological, chemical, transportation, electronic, communication, manufacturing and power systems [5]. Time-delay and multi-delay control systems are therefore very important classes of systems whose control and optimization have been of interest to many investigators [2-6]. Orthogonal functions (OFs) and polynomial series have received considerable attentions in dealing with various problems of dynamic systems. Much progress has been made towards the solution of delay systems. The approach is that of converting the delay-differential equation govering the dynamical systems to an algebraic form through the use of an operational matrix of integration . The matrix can be uniquely determined based on the particular OFs. Special attentions has been given to applications of Walsh functions [3], block-pulse functions[12], Laguerre polynomials [6], Legendre polynomials [7], Chebyshev polynomials[4] and Fourier series [9]. The available sets of OFs can be divided into three classes. The first includes a set of piecewise constant basis functions (PCBFs) (e.g. Walsh, block-pulse, etc.). The second consists of a set of orthogonal polynomials (OPs) (e.g. Laguerre, Legendre, Chebyshev, etc). The third is the widely used set of sine-cosine functions (SCFs) in the form of Fourier series. In this paper we use Fourier series method to solve time delay control systems. The method consists of reducing the delay problem to a set of algebraic equations by first expanding the candidate function as a Fourier series with unknown coefficients. These Fourier series are first introduced. The operational matrices of integration, delay and product are given. These matrices are then used to evaluate the coefficients of the Fourier series for the solution of time delay control systems.

II. Fourier series and their properties:

2.1. Expansion by Fourier series:

A function $f(t)$ belongs to the apace $L^2[0, L]$ may be expanded by Fourier series as follows[11]:

$$f(t) = a_0 + \sum_{n=1}^{\infty}\{a_n cos\frac{2n\pi t}{L} + a_n^* sin\frac{2n\pi t}{L}\}, \tag{1}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(t)dt,$$

$$a_n = \frac{2}{L}\int_0^L f(t)cos\frac{2n\pi t}{L}dt, \qquad n = 1,2,3,\ldots,$$

$$a_n^* = \frac{2}{L}\int_0^L f(t)sin\frac{2n\pi t}{L}dt, \qquad n = 1,2,3,\ldots.$$

By truncating the series (1) up to $(2r+1)$th term we can abtain an approximation for $f(t)$ as follows:

$$f(t) \approx a_0 + \sum_{n=1}^{r}\{a_n \phi_n(t) + a_n^* \phi_n^*(t)\} = A^T \phi(t) \tag{2}$$

where

$$A = [a_0, a_1, a_2, \ldots, a_r, a_1^*, a_2^*, \ldots, a_r^*]^T,$$
$$\phi(t) = [\phi_0(t), \phi_1(t), \phi_2(t), \ldots, \phi_r(t), \phi_1^*(t), \phi_2^*(t), \ldots, \phi_r^*(t)]^T, \tag{3}$$

and

$$\phi_n(t) = cos\frac{2n\pi t}{L}, \qquad n = 0,1,2,\ldots,r,$$
$$\phi_n^*(t) = sin\frac{2n\pi t}{L}, \qquad n = 1,2,\ldots,r.$$

It can be easily seen that the elements of $\phi(t)$ in interval $(0, L)$ are orthogonal.

2.2. Operational matrices of integration, product and delay:

The integration of $\phi(t)$ in (3) can be approximated by $\phi(t)$ as follows:

$$\int_0^t \phi(s)ds \approx P\phi(t)$$

where P is the operational matrix of integration of order $(2r+1)\times(2r+1)$ and is given by[8,10]:

$$P = L\begin{pmatrix}
\frac{1}{2} & 0 & 0 & \ldots & 0 & 0 & \frac{-1}{\pi} & \frac{-1}{2\pi} & \ldots & \frac{-1}{(r-1)\pi} & \frac{-1}{r\pi} \\
0 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{2\pi} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \frac{1}{2(r-1)\pi} & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{2r\pi} \\
\frac{1}{2\pi} & \frac{-1}{2\pi} & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\frac{1}{4\pi} & 0 & \frac{-1}{4\pi} & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{2r\pi} & 0 & 0 & \ldots & 0 & \frac{-1}{2r\pi} & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}.$$

Furthermore we have:

$$\phi(t)\phi^T(t) = \begin{pmatrix}
\phi_0^2 & \phi_0\phi_1 & \ldots & \phi_0\phi_r & \phi_0\phi_1^* & \ldots & \phi_0\phi_r^* \\
\phi_1\phi_0 & \phi_1^2 & \ldots & \phi_1\phi_r & \phi_1\phi_1^* & \ldots & \phi_1\phi_r^* \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
\phi_r\phi_0 & \phi_r\phi_1 & \ldots & \phi_r^2 & \phi_r\phi_1^* & \ldots & \phi_r\phi_r^* \\
\phi_1^*\phi_0 & \phi_1^*\phi_1 & \ldots & \phi_1^*\phi_r & \phi_1^{*2} & \ldots & \phi_1^*\phi_r^* \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
\phi_r^*\phi_0 & \phi_r^*\phi_1 & \ldots & \phi_r^*\phi_r & \phi_r^*\phi_1^* & \ldots & \phi_r^{*2}
\end{pmatrix}, \tag{4}$$

one can esealy show that $\phi(t)\phi^T(t)A = \tilde{A}\phi(t)$ where $\tilde{A}$ is called the product operational matrix for the so called vector A in (3) and is given:

$$\tilde{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_r & a_1^* & a_2^* & \dots & a_r^* \\ \frac{1}{2}a_1 & a_0+\frac{1}{2}a_2 & \frac{1}{2}(a_1+a_3) & \dots & \frac{1}{2}a_{r-1} & \frac{1}{2}a_2^* & \frac{1}{2}(a_1^*+a_3^*) & \dots & \frac{1}{2}a_{r-1}^* \\ \frac{1}{2}a_2 & \frac{1}{2}(a_1+a_3) & a_0+\frac{1}{2}a_4 & \dots & \frac{1}{2}a_{r-2} & \frac{1}{2}(a_3^*-a_1^*) & \frac{1}{2}a_4^* & \dots & \frac{1}{2}a_{r-2}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}a_r & \frac{1}{2}a_{r-1} & \frac{1}{2}a_{r-2} & \dots & a_0 & -\frac{1}{2}a_{r-1}^* & -\frac{1}{2}a_{r-2}^* & \dots & 0 \\ \frac{1}{2}a_1^* & \frac{1}{2}a_2^* & \frac{1}{2}(a_3^*-a_1^*) & \dots & -\frac{1}{2}a_{r-1}^* & a_0-\frac{1}{2}a_2 & \frac{1}{2}(a_1-a_3) & \dots & \frac{1}{2}a_{r-1} \\ \frac{1}{2}a_2^* & \frac{1}{2}(a_3^*+a_1^*) & \frac{1}{2}a_4^* & \dots & -\frac{1}{2}a_{r-2}^* & \frac{1}{2}(a_1-a_3) & a_0-\frac{1}{2}a_4 & \dots & \frac{1}{2}a_{r-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}a_r^* & \frac{1}{2}a_{r-1}^* & \frac{1}{2}a_{r-2}^* & \dots & 0 & \frac{1}{2}a_{r-1} & \frac{1}{2}a_{r-2} & \dots & a_0 \end{pmatrix}_{(2r+1)\times(2r+1)}.$$

By integrating of (4) and considering the orthogonality components of $\phi(t)$ we have:

$$E = \int_0^L \phi(t)\phi^T(t)dt, \quad (5)$$

where

$$E = L\begin{pmatrix} 1 & & & & \\ & 1/2 & & 0 & \\ & & 1/2 & & \\ & 0 & & \ddots & \\ & & & & 1/2 \end{pmatrix}_{(2r+1)\times(2r+1)}. \quad (6)$$

Now we are going to derive operational delay matrix. From calculus we know that:

$$sin(\alpha \pm \beta) = sin\alpha cos\beta \pm cos\alpha sin\beta,$$
$$cos(\alpha \pm \beta) = cos\alpha cos\beta \mp sin\alpha sin\beta,$$

so if τ is any delay(or lag) for the functions $\phi(t)$ and $\phi^*(t)$ then we have:

$$\phi_n(t-\tau) = cos(\frac{2n\pi t}{L})\phi_n(t) + sin(\frac{2n\pi t}{L})\phi_n^*(t),$$
$$\phi_n^*(t-\tau) = cos(\frac{2n\pi t}{L})\phi_n^*(t) - sin(\frac{2n\pi t}{L})\phi_n(t).$$

Now it is esealy to show that $\phi(t-\tau) = D_\tau\phi(t)$, where D_τ is delay operational matrix and have the following form:

$$D_\tau = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & cos(\frac{2\pi\tau}{L}) & 0 & \dots & 0 & sin(\frac{2\pi\tau}{L}) & 0 & \dots & 0 \\ 0 & 0 & cos(\frac{4\pi\tau}{L}) & \dots & 0 & 0 & sin(\frac{4\pi\tau}{L}) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & cos(\frac{2r\pi\tau}{L}) & 0 & 0 & \dots & sin(\frac{2r\pi\tau}{L}) \\ 0 & -sin(\frac{2\pi\tau}{L}) & 0 & \dots & 0 & cos(\frac{2\pi\tau}{L}) & 0 & \dots & 0 \\ 0 & 0 & -sin(\frac{4\pi\tau}{L}) & \dots & 0 & 0 & cos(\frac{4\pi\tau}{L}) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -sin(\frac{2r\pi\tau}{L}) & 0 & 0 & \dots & cos(\frac{2r\pi\tau}{L}) \end{pmatrix}.$$

III. Problem statement

Consider the following quadratic time-independent delay control system:

$$\min \qquad J = \frac{1}{2}x^T(t_f)Sx(t_f) + \frac{1}{2}\int_0^{t_f}\{x^T(t)Qx(t) + u^T(t)Ru(t)\}dt \qquad (7)$$

$$s.t \qquad \dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t) + Du(t-\tau),\ 0 \le t \le L, \qquad (8)$$

$$x(0) = x_0, \qquad (9)$$

$$x(t) = \theta(t),\ -\tau \le t < 0, \qquad (10)$$

$$u(t) = \psi(t),\ -\tau \le t < 0, \qquad (11)$$

(For the time being, assume that the matrices A, B, C, D are constant but the result can be extended to time varying systems by appropriate changes) where R is symmetric positive definite and Q, S are positive semidefinite matrices, $x(t) \in R^l$, $u(t) \in R^q$ are state and control vectores respectively and A, B, C, D are matrices of appropriate dimensions, x_0 is a constant specified vector, and $\theta(t), \psi(t)$ are arbitrary known functions. The problem is to find $x(t)$ and $u(t)$, $0 \le t \le L$, satisfying (8)-(11) while minimizing (7).

Assume that

$$x(t) = X^T\phi(t), \qquad where\ \ X = [x_0, x_1, x_2, \dots, x_r, x_1^*, x_2^*, \dots, x_r^*]^T,$$

$$u(t) = U^T\phi(t), \qquad where\ \ U = [u_0, u_1, u_2, \dots, u_r, u_1^*, u_2^*, \dots, u_r^*]^T,$$

$$x(0) = X_0^T\phi(t), \qquad where\ \ X_0 = [x(0), 0, 0, \dots, 0, 0, \dots, 0]^T.$$

Due to (10), in $-\tau \le t \le 0$ we have $x(t) = \theta(t)$ so for $0 \le t \le \tau$ and consequently for $-\tau \le t - \tau \le 0$ it is true to have $x(t-\tau) = \theta(t-\tau) = G^T\phi(t),$ where G^T is the Fourier series coefficient of $\theta(t-\tau)$. Now we have

$$x(t-\tau) = \begin{cases} \theta(t-\tau) = G^T\phi(t), & 0 \le t \le \tau \\ X^T\phi(t-\tau) = X^T D_\tau\phi(t), & \tau \le t \le L \end{cases} \qquad (12)$$

and by the same approach with respect to (11)

$$u(t-\tau) = \begin{cases} \psi(t-\tau) = H^T\phi(t), & 0 \le t \le \tau \\ U^T\phi(t-\tau) = U^T D_\tau\phi(t), & \tau \le t \le L \end{cases} \qquad (13)$$

The integration of (8) from 0 to t and using of (9) gives

$$\int_0^t \dot{x}(s)ds = A\int_0^t x(s)ds + B\int_0^t x(s-\tau)ds + C\int_0^t u(s)ds + D\int_0^t u(s-\tau)ds, \tag{14}$$

or equivalently

$$x(t) - x(0) = A\int_0^t x(s)ds + B\int_0^\tau x(s-\tau)ds + B\int_\tau^t x(s-\tau)ds$$
$$+ C\int_0^t u(s)ds + D\int_0^\tau u(s-\tau)ds + D\int_\tau^t u(s-\tau)ds. \tag{15}$$

Now, from (12) and (13) we have:

$$\int_\tau^t x(s-\tau)ds = \int_\tau^t X^T D_\tau \phi(s)ds = \int_0^t X^T D_\tau \phi(s)ds - \int_0^\tau X^T D_\tau \phi(s)ds$$
$$= X^T D_\tau P\phi(t) - X^T D\tau Z\phi(t)$$
$$\int_\tau^t u(s-\tau)ds = \int_\tau^t U^T D_\tau \phi(s)ds = \int_0^t U^T D_\tau \phi(s)ds - \int_0^\tau U^T D_\tau \phi(s)ds$$
$$= U^T D_\tau P\phi(t) - U^T D_\tau Z\phi(t)$$
$$\int_0^\tau \phi(t)dt = Z\phi(t) \tag{16}$$

where

$$Z = \begin{pmatrix} \tau & 0 & \dots & 0 & 0 & \dots & 0 \\ \frac{L}{2\pi} sin(\frac{2\pi\tau}{L}) & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{L}{2r\pi} sin(\frac{2r\pi\tau}{L}) & 0 & \dots & 0 & 0 & \dots & 0 \\ \frac{L}{2\pi}(1-cos(\frac{2\pi\tau}{L})) & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{L}{2r\pi}(1-cos(\frac{2r\pi\tau}{L})) & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{17}$$

Thus (15) reduces to

$$X^T\phi(t) - X_0^T\phi(t) = AX^T P\phi(t) + BG^T Z\phi(t) + BX^T D_\tau P\phi(t) - BX^T D_\tau Z\phi(t)$$
$$+ CU^T P\phi(t) + DH^T Z\phi(t) + DU^T D_\tau P\phi(t) - DU^T D_\tau Z\phi(t). \tag{18}$$

By deleting $\phi(t)$ from both sides of (18) and ordering we conclude that:

$$C^* = X^T - X_0^T - AX^T P - BG^T Z - BX^T D_\tau P + BX^T D_\tau Z$$
$$- CU^T P - DH^T Z - DU^T D_\tau P + DU^T D_\tau Z = 0 \tag{19}$$

By substituting the Fourier series in (J) we have

$$J = \frac{1}{2}\phi^T(t_f)XSX^T\phi(t_f) + \frac{1}{2}\int_0^{t_f}\{\phi^T(t)XQX^T\phi(t) + \phi^T(t)URU^T\phi(t)\}dt \tag{20}$$

The optimal control problem has now been reduced to a parameter optimization problem which can be stated as follows. Find X and U so that $J(X,U)$ is minimized subject to the constraints in Eq. (19).

Let

$$J^*(X,U,\lambda) = J(X,U) + \lambda^T C^* \tag{21}$$

where the vector λ represents the unknown Lagrange multipliers, then the necessary conditions for stationary are given by

$$\frac{\partial}{\partial X} J^*(X,U,\lambda) = 0$$

$$\frac{\partial}{\partial U} J^*(X,U,\lambda) = 0$$

$$\frac{\partial}{\partial \lambda} J^*(X,U,\lambda) = 0 \tag{22}$$

Remark1. Note that if delays in state and control vectors aren't the same, then solving the system is the same as previous but we have two or more delay operational matrices.

IV. Illustrative Examples:

In this section two examples are given to demonstrate the applicability, efficiency and accuracy of our proposed method.

4.1. Example 1:

Consider the following system [13]

$$\min \qquad J = \frac{1}{2}\int_0^2 \{x^2(t) + u^2(t)\}dt \tag{23}$$

$$s.t \qquad \dot{x}(t) = x(t-1) + u(t), \ \ 0 \le t \le 2, \tag{24}$$

$$x(t) = 1, \ \ -1 \le t \le 0.$$

Here, we solve the same problem by using of Fourier series. Note that in third example delay is applied on state only, and $\tau = 1$. Suppose that

$$x(t) = X^T\phi(t), \quad u(t) = U^T\phi(t), \quad x(0) = X_0^T\phi(t),$$

where $X^T, U^T, \phi(t), X_0$ are defined previously. If we integrate (24) from 0 to t and use (12)-(13)we have

$$\int_0^t \dot{x}(s)ds = \int_0^t x(s-1)ds + \int_0^t u(s)ds$$

$$= \int_0^1 x(s-1)ds + \int_1^t x(s-1)ds + \int_0^t u(s)ds,$$

and we have

$$\int_1^t x(s-1)ds = \int_0^t x(s-1)ds - \int_0^1 x(s-1)ds,$$

so we obtain

$$X^T\phi(t) - X_0^T\phi(t) = G^T Z\phi(t) + X^T D_\tau P\phi(t) - X^T D_\tau Z\phi(t) + U^T P\phi(t).$$

By deleting $\phi(t)$ from both sides and ordering we conclude that:

$$C^* = X^T - X_0^T - G^T Z - X^T D_\tau P + X^T D_\tau Z - U^T P = 0$$

By substitution the Fourier series in (23) for J we have

$$J = \int_0^1 \{X^T\phi(t)\phi^T(t)X + U^T\phi(t)\phi^T(t)U\}dt$$

$$= X^T EX + U^T EU,$$

where E is defined in (6). Thus we have redused the system as follows

$$\min \qquad J = X^T EX + U^T EU$$

$$S.to \qquad C^* = X^T - X_0^T - G^T Z - X^T D_\tau P + X^T D_\tau Z - U^T P = 0$$

Approximate solutions of $x(t), u(t)$ with $r = 25$ are shown in Fig.1. The value of J is 1.62421313.

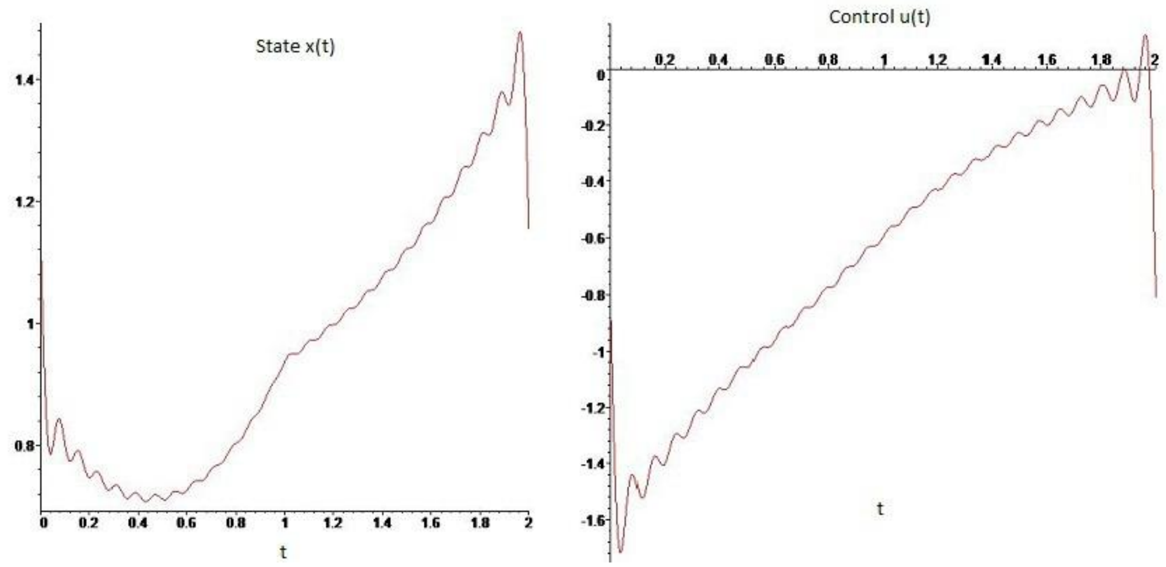

Figure 1: State vector x(t) and control u(t) for r=25 in Example 1

4.2. Example 2:

Consider the following system [8,13]

$$\min \qquad J = \frac{1}{2}\int_0^1 \{x^2(t) + \frac{1}{2}u^2(t)\}dt \tag{25}$$

$$s.t \qquad \dot{x}(t) = -x(t) + x(t-1/3) + u(t) - \frac{1}{2}u(t-2/3),\ \ 0 \le t \le 1, \tag{26}$$

$$x(t) = 1,\ \ -1 \le t \le 0,$$

$$u(t) = 0,\ \ -2/3 \le t < 0.$$

Here we have different delays in state($\tau_1 = 1/3$) and control($\tau_2 = 2/3$). Suppose that

$$x(t) = X^T\phi(t), \quad u(t) = U^T\phi(t), \quad x(0) = X_0^T\phi(t),$$

where $X^T, U^T, \phi(t), X_0$ are defined previously. If we integrate (26) from 0 to t and use (12)-(13) we have

$$\int_0^t \dot{x}(s)ds = -\int_0^t x(s)ds + \int_0^t x(s-\frac{1}{3})ds + \int_0^t u(s)ds - \frac{1}{2}\int_0^t u(s-\frac{2}{3})ds$$

$$x(t) - x(0) = -\int_0^t x(s)ds + \int_0^{\frac{1}{3}} x(s-\frac{1}{3})ds + \int_{\frac{1}{3}}^t x(s-\frac{1}{3})ds + \int_0^t u(s)ds$$

$$-\frac{1}{2}\int_0^{\frac{2}{3}} u(s-\frac{2}{3})ds - \frac{1}{2}\int_{\frac{2}{3}}^t u(s-\frac{2}{3})ds,$$

so we obtain

$$X^T\phi(t) - X_0^T\phi(t) = -X^T P\phi(t) + X_0^T Z_1\phi(t) + X^T D_{\tau_1} P\phi(t) - X^T D_{\tau_1} Z_1\phi(t)$$

$$+U^T P\phi(t) - \frac{1}{2}U^T D_{\tau_2} P\phi(t) + \frac{1}{2}U^T D_{\tau_2} Z_2\phi(t)$$

By deleting $\phi(t)$ from both sides and ordering we conclude that:

$$C^* = X^T - X_0 + X^T P - X_0^T Z_1 - X^T D_{\tau_1} P + X^T D_{\tau_1} Z_1$$

$$-U^T P + \frac{1}{2}U^T D_{\tau_2} P - \frac{1}{2}U^T D_{\tau_2} Z_2 = 0,$$

By substitution the Fourier series in (23) for J we have

$$J = \frac{1}{2}\int_0^1 \{X^T\phi(t)\phi^T(t)X + \frac{1}{2}U^T\phi(t)\phi^T(t)U\}dt$$
$$= \frac{1}{2}(X^T EX + \frac{1}{2}U^T EU),$$

where E was defined in (6). now we have redused system as follows

$$\min \qquad J = \frac{1}{2}(X^T EX + \frac{1}{2}U^T EU)$$
$$s.t \qquad C^* = X^T - X_0 + X^T P - X_0^T Z_1 - X^T D_{\tau_1} P + X^T D_{\tau_1} Z_1$$
$$-U^T P + \frac{1}{2}U^T D_{\tau_2} P - \frac{1}{2}U^T D_{\tau_2} Z_2 = 0$$

Approximate solutions of $x(t), u(t)$ with $r = 25$ are shown in Fig.2. the value of J is 0.35944042 (In [3], the value of J is 0.3731).

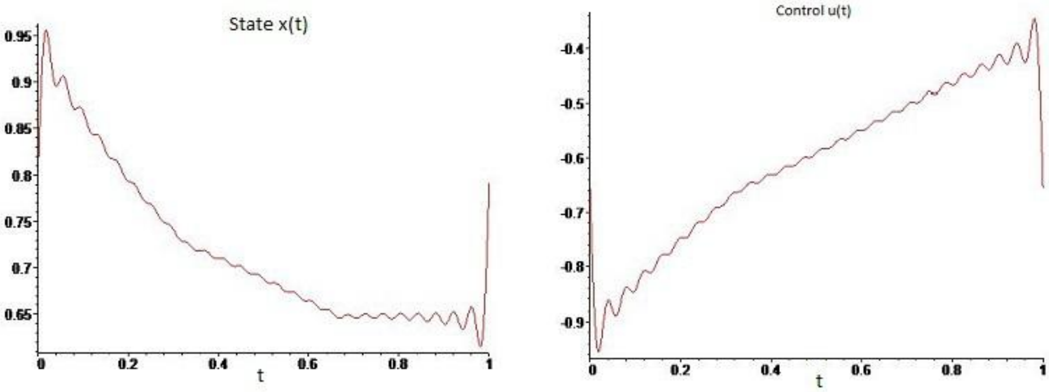

Figure 2: State vector x(t) and control u(t) for r=25 in Example 2

4.3. Example 3:

Consider the following system[1,3]

$$\min \qquad J = \int_0^3 \{x^2(t) + u^2(t)\}dt \tag{27}$$
$$s.t \qquad \dot{x}(t) = x(t-1)u(t-2), \tag{28}$$
$$x(t) = 1, \ -1 \le t \le 0,$$
$$u(t) = 0, \ -2 \le t \le 0,$$

Here, the delays are $\tau_1 = 1$ for state and $\tau_2 = 2$ for control vectors, respectively. Suppose that

$$x(t) = X^T\phi(t), \quad u(t) = U^T\phi(t), \quad x(0) = X_0^T\phi(t),$$

where $X^T, U^T, \phi(t), X_0$ are defined previously. With respect to (12) and (13) we have

$$x(t-1) = \begin{cases} \theta(t-1) = G^T\phi(t) = X_0\phi(t), & 0 \le t \le 1 \\ X^T\phi(t-1) = X^T D_{\tau_1}\phi(t), & 1 \le t \le 3 \end{cases}$$

$$u(t-2) = \begin{cases} \psi(t-2) = 0, & 0 \le t \le 2 \\ U^T\phi(t-2) = U^T D_{\tau_2}\phi(t), & 2 \le t \le 3 \end{cases}$$

If we integrate (28) from 0 to t then

$$\int_0^t \dot{x}(s)ds = \int_0^t x(s-1)u(s-2)ds$$
$$= \int_0^1 x(s-1)u(s-2)ds + \int_1^2 x(s-1)u(s-2)ds + \int_2^t x(s-1)u(s-2)ds$$

By substitution the Fourier series and simplifying, we obtain

$$X^T\phi(t) - X_0^T\phi(t) = \int_0^1 [X_0^T\phi(s)0]ds + \int_1^2 [X^T D_{\tau_1}\phi(s)0]ds + \int_2^t [X^T D_{\tau_1}\phi(s)U^T D_{\tau_2}\phi(s)]ds,$$

since we have $u(t-2) = 0,\ \ 0 \le t \le 2$. Now we have

$$X^T\phi(t) - X_0^T\phi(t) = \int_2^t [X^T D_{\tau_1}\phi(s)U^T D_{\tau_2}\phi(s)]ds$$

$$= \int_0^t [X^T D_{\tau_1}\phi(s)U^T D_{\tau_2}\phi(s)]ds - \int_0^2 [X^T D_{\tau_1}\phi(s)U^T D_{\tau_2}\phi(s)]ds$$

$$= X^T D_{\tau_1}\int_0^t [\phi(s)\phi^T(s)D_{\tau_2}^T U]ds - X^T D_{\tau_1}\int_0^2 [\phi(s)\phi^T(s)D_{\tau_2}^T U]ds \tag{29}$$

Assume that $D_{\tau_2}^T U = U_1$, thus (29) reduces to

$$C^* = X^T - X_0^T - X^T D_{\tau_1}\widetilde{U}_1 P + X^T D_{\tau_1}\widetilde{U}_1 Z = 0 \tag{30}$$

By substitution the Fourier series for J we have

$$J = \int_0^1 \{X^T\phi(t)\phi^T(t)X + U^T\phi(t)\phi^T(t)U\}dt = X^T EX + U^T EU, \tag{31}$$

where E is defined in (6). So the delay optimal control (27-28) now is reduced to the following optimazation problem

$$\min \qquad J = X^T EX + U^T EU$$

$$s.t \qquad C^* = X^T - X_0^T - X^T D_{\tau_1}\widetilde{U}_1 P + X^T D_{\tau_1}\widetilde{U}_1 Z = 0$$

Approximate solutions of $x(t), u(t)$ with $r = 30$ are shown in Fig.3. the value of J is 2.3496 while the exact value is $\dfrac{3e^2+1}{e^2+1}$.

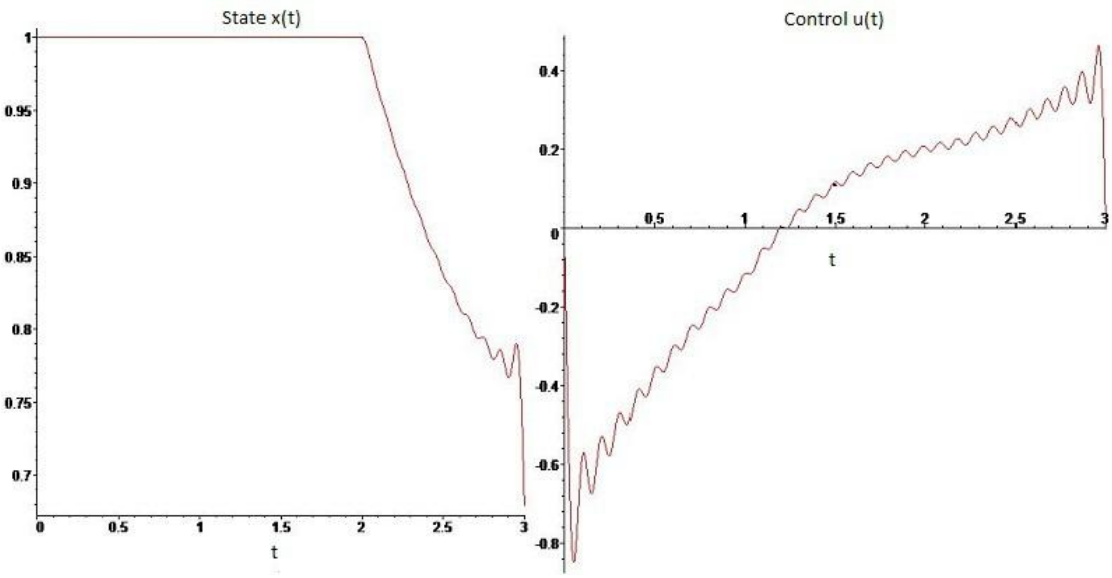

Figure 3: State vector x(t) and control u(t) for r=30 in Example 3

V. Conclusions

Using Fourier series, a simple and computational method for solving time delay optimal problems is considered. The method is based upon reducing a nonlinear time delay optimal control problem to an nonlinear programming problem. The unity of the function of orthogonality for Fourier series and the simplicity of applying delays in Fouries series are great merits that make the approach very attractive and easy to use.
Although the method is simple, by solving various examples, accuracy in comparison of the other methods can be found.

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