RESEARCH ARTICLE OPEN ACCESS

Determination of the Optimal Guardbanding to Ensure Acceptable Risk Decision in the Declaration of Conformity

Hicham Mezouara[1], latifa Dlimi[2], Abdelouahhab Salih[3], Mohamed Afechcar[4]
[1](Faculty of Science Department of Physics Ibn Tofail University Kenitra, Morocco)
[2](Faculty of Science Department of Physics Ibn Tofail University Kenitra, Morocco)
[3](PES Génie Mécanique LMD- E.N.S.E.T. B.P. 6207 Rabat – Maroc Responsable de l'Equipe de Recherche Instrumentation, Mesures et Essais Auditeur Technique attaché au MCI, Consultant de l'ANPME)
[4](National Center for Studies and Road Research CNER/DRCR BP 6226, Rabat-Instituts Rabat, Morocco)

ABSTRACT
This article proposes a mathematical optimization procedure designed to generate an economical measurement model for determining the risk of decision error (customer risk). The model includes an optimal guardbanding to reduce the impacts of measurement errors .A mathematical model is provided as an example, and conclusions are drawn.
***Keywords**:* guardbanding, consumer rick, measurement errors.

I. Introduction

In many manufacturing industries, measurement procedures associated with the inspection of products have become an integral part of quality improvement and control. Even so, some measurement errors are inevitable due to changes in operators and/or devices, regardless of how carefully the measurement procedures are designed or maintained .There have been many research efforts to reduce the impact of measurement errors and to improve quality control. The most immediate approach may be to control measurement error by the selection of an optimal guardbanding [1, 2, 3].

II. Determination of Guardbanding Width

Measurement precision may be improved by reducing measurement variability. Chandra and schall [4] proposed the use of repeated measurements to reduce measurement variability, the average of these repeated measurements is used to determine the conformance of a product to the specifications.
Let *X* be the actual value of the quality characteristic of interest, which is normally distributed with a mean of μ and a variance of σ_X^2. If we denote the measured value from a single measurement as *Y*, let us further assume that the conditional distribution of *Y*, given that $X = x$, is a normal distribution with a mean of *x* and a variance $\sigma_{y|x}^2$.

Suppose that *n* measurements are repeatedly taken and each measurement has the same variability. Letting $\overline{Y}$ be the average of *n* measurements, it is apparent that the conditional distribution of $\overline{Y}$, given that $X = x$, is a normal distribution with a mean of *x* and a variance of

$\sigma_{\overline{y}|x}^2$,where $\sigma_{\overline{y}|x}^2 = \dfrac{\sigma_{y|x}^2}{n}$. As a means of reducing the impact of measurement errors, the use of guard bands has been widely implemented since being introduced by Eagle [5].

In many practical situations, a false acceptance of defects incurs much larger economic penalties than a false rejection of conforming products. From this perspective, many manufacturers impose a guardbanding to help minimize the penalty associated with false acceptance, at the cost of an increased risk of false rejection. The effects of a guardbanding are depicted in Fig. 1, where *L* and *U* represent the lower and upper specification limits. The large curve represents the density curve of the actual value of the quality characteristic, *X*, while the small curve represents the density curve of the average measurements given the actual value, $(\overline{Y} \mid X)$. It can be observed that the probability of false acceptance decreases by imposing the guardbanding (Fig. 1 (a)) while the risk associated with false rejection increases (Fig. 1(b)). It is a current practice to set the guardbanding based on engineering experiences or on a trial-and-error basis.

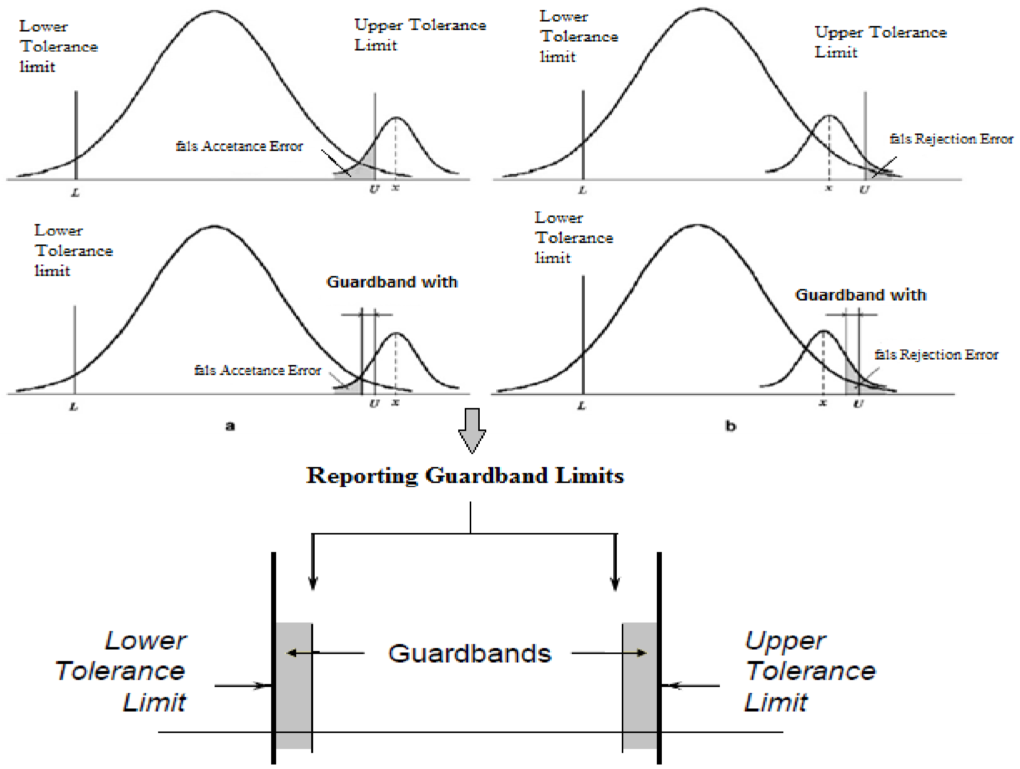

Figure 1.Measurement errors with and without guard band.(a) False acceptance error with and without guard band.(b) False rejection error with and without guard band.

To meet customer requirements and to avoid the high cost of passing bad product to customers, the consumer's risk should not exceed a specified value. In this case, the guardbanding υ and ω are set inside the specification limits L and U. Let $\upsilon = L+\varepsilon_L$ and $\omega = U- \varepsilon_U$, then ε_L and ε_U are positive. On the other hand, if the consumer's risk exceeds the specified value, then ε_L or ε_U may be negative. In this paper, we focus on meeting customer's requirements so that ε_L and ε_U are positive. In general, the consumer's risk should be much lower than the producer's risk because the cost of letting bad product get to consumers is usually much higher than the cost of rejecting good product. The difference between the product tolerance and the length of the guardbanding interval is (U - L) - (υ - ω) = ($\varepsilon_L + \varepsilon_U$). The optimal guardbanding interval (υ, ω) or the pair (ε_L , ε_U) with the smallest ($\varepsilon_L + \varepsilon_U$) can be determined so that $\beta \leq \beta 0$ where β0 preset level and the expression for β are given by Eq (20).

III. Study of Customer Risk

The customer risk is the percentage of non-conforming products that are delivered, and accepted by the customer. it is calculated as the product of the probability of making a " non-conforming " product (property of the production process) by the (conditional) probability of measuring "compliant" (i.e. in the tolerance).

Let ε_L and ε_U denote the widths of guardbanding associated with the lower and upper specification limits , respectively. For the simplicity of notation, $\upsilon = L+\varepsilon_L$ and $\omega = U- \varepsilon_U$. Hereafter, υ and ω are referred to as the lower and upper inspection limits, respectively. Since the conformance of a product is determined on the basis of repeated measurements, a product passes the inspection and is shipped to the customer if $\overline{y}\epsilon[\upsilon,\omega]$ and $R_C = \beta$ represent the customer risk (the risk of delivering a product that is intrinsically unacceptable but that is accepted by the control means),i.e.,$\overline{y}\,\epsilon[\upsilon,\omega]$ and $X\notin[L,U]$.

The expected cost by falsely accepting a defect, denoted by $\beta^{`}$, is the conditional probability that a product will be accepted given that it is defective. it is then given by

$$\beta' = \frac{\beta}{1 - P(x \in [-L, L])} \qquad (1)$$

Where $P(x\in[-L, L])$ is the probability that the value being measured lies in $[-L, L]$.

$$R_C = \beta = \int_{\upsilon}^{\omega}\int_{U}^{+\infty} h(x,\overline{y})\, dx d\overline{y} + \int_{\upsilon}^{\omega}\int_{-\infty}^{L} h(x,\overline{y})\, dx d\overline{y}, \qquad (2)$$

Where$h(x,\overline{y})$ is the joint density function of X and $\overline{Y}$. It can easily be shown that X and $\overline{Y}$ jointly follow a bivariate normal distribution with a mean vector of (μ, μ) and a variance–covariance matrix of Σ given by

$$\Sigma = \begin{bmatrix} \mathrm{var(x)} & \mathrm{cov}(x,\bar{y}) \\ \mathrm{cov}(x,\bar{y}) & \mathrm{var}(\bar{y}) \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_{\bar{y}}^2 \end{bmatrix} \quad (3)$$

Where $\sigma_{\bar{y}}^2$ represents the variance of the marginal distribution of $\bar{Y}$, and $\sigma_{\bar{y}}^2 = \sigma_x^2 + \sigma_{\bar{y}|x}^2$. Note that γ, the correlation coefficient of X and $\bar{Y}$, is defined as

$$\gamma = \frac{\mathrm{cov}(X,\bar{Y})}{\sqrt{\mathrm{var}(X)\times \mathrm{var}(\bar{Y})}} = \frac{\sigma_x}{\sigma_{\bar{y}}} \quad , \quad (4)$$

The marginal distribution X and $\bar{Y}$, assume that the density $h(x,\bar{y})$ exists, we denote by $g(\bar{y}\,|\,x)$ and $f(x)$ [6,7,8].

And $h(x,\bar{y}) = g(\bar{y}\,|\,x)\, f(x)$, (5)

where

$$g(\bar{y}\,|\,x) = \frac{1}{\sqrt{2\pi}\,\sigma_{\bar{y}|x}} e^{-\frac{(\bar{y}-x)^2}{2\sigma_{\bar{y}|x}^2}} \quad , \quad (6)$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma x} e^{-\frac{(x-\mu)^2}{2\sigma_x^2}} \quad , \quad (7)$$

The first double integral in Eq. (2) can be written as

$$\int_{\upsilon}^{\omega}\int_{U}^{\infty} h(x,\bar{y})dxd\bar{y} = \int_{U}^{\infty}\int_{\upsilon}^{\omega} h(x,\bar{y})d\bar{y}dx$$

$$= \int_{U}^{\infty}[\int_{\upsilon}^{\omega} g(\bar{y}\,|\,x)d\bar{y}]f(x)dx \quad , \quad (8)$$

where $g(\bar{y}\,|\,x)$ and $f(x)$ are the conditional distribution of $\bar{Y}$ since $X=x$ and the marginal distribution of X, respectively. Noting that

$$\bar{Y}\Big|_{X=x} \approx N(x,\sigma_{\bar{y}|x}^2) \text{ and}$$

$$X \approx N(\mu,\sigma_x^2) \quad , \quad (9)$$

Let $z = \frac{\bar{y}-x}{\sigma_{\bar{y}|x}}$, and $\upsilon \le \bar{y} \le \omega$ is then given

$$\frac{\upsilon-x}{\sigma_{\bar{y}|x}} \le z \le \frac{\omega-x}{\sigma_{\bar{y}|x}} \quad , \quad (10)$$

And $\lambda = \frac{x-\mu}{\sigma_x}$, with $U \le x \le +\infty$ is then given

$$\frac{U-\mu}{\sigma_x} \le \lambda \le +\infty \quad , \quad (11)$$

Can be written as Eq.(8):

$$\int_{U}^{\infty}[\int_{\upsilon}^{\omega} g(\bar{y}\,|\,x)d\bar{y}]f(x)dx = \int_{U}^{\infty}[\int_{\frac{\upsilon-x}{\sigma_{\bar{y}|x}}}^{\frac{\omega-x}{\sigma_{\bar{y}|x}}} \phi(z)dz]f(x)dx$$

$$= \int_{U}^{\infty}[\Phi(\frac{\omega-x}{\sigma_{\bar{y}|x}}) - \Phi(\frac{\upsilon-x}{\sigma_{\bar{y}|x}})]f(x)dx$$

$$= \int_{U}^{\infty}\Phi(\frac{\omega-x}{\sigma_{\bar{y}|x}})f(x)dx - \int_{U}^{\infty}\Phi(\frac{\nu-x}{\sigma_{\bar{y}|x}})f(x)dx$$

$$= \int_{\frac{U-\mu}{\sigma_x}}^{\infty}\Phi(\frac{\omega-(\mu+\sigma_x\lambda)}{\sigma_{\bar{y}|x}})\phi(\lambda)d\lambda - \int_{U-\mu/\sigma_x}^{\infty}\Phi(\frac{\nu-(\mu+\sigma_x\lambda)}{\sigma_{\bar{y}|x}})\phi(\lambda)d\lambda$$

(12)

Here, $\Phi(.)$ and $\varphi(.)$ represent the cumulative distribution and probability density functions of the standard normal distribution, respectively [9]. Using the following identity,

$$\int_{K}^{\infty}\Phi(a+b\lambda)\phi(\lambda)d\lambda = BVN(\frac{a}{\sqrt{1+b^2}};-K;\frac{b}{\sqrt{1+b^2}}) \quad (13)$$

where $BVN(\alpha,\beta,\rho)$ represents a function with two variables standard normal distribution with a correlation coefficient of ρ, which is defined by

$$BVN(\alpha,\beta,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\alpha}\int_{-\infty}^{\beta}\exp(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)})dxdy \quad (14)$$

With

$$\varphi(x) = \frac{d\Phi(x)}{dx} \quad , \quad (15)$$

$$\Phi(x) = \int_{-\infty}^{h}\varphi(x)dx \quad , \quad (16)$$

and

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2}) \; , \quad (17)$$

Eq.(12) can be simplified to

$$\int_{\upsilon}^{\omega}\int_{U}^{\infty} h(x,\bar{y})dxd\bar{y} = BVN(\frac{\omega-\mu}{\sqrt{\sigma_x^2+\sigma_{\bar{y}|x}^2}};-\frac{U-\mu}{\sigma_x};-\gamma) - BVN(\frac{\upsilon-\mu}{\sqrt{\sigma_x^2+\sigma_{\bar{y}|x}^2}};-\frac{U-\mu}{\sigma_x};-\gamma) \; , \quad (18)$$

Similarly, it can be shown that the second integral in Eq. (2) becomes:

$$\int_{\upsilon}^{\omega}\int_{-\infty}^{L} h(x,\bar{y})dxd\bar{y} = \Phi(\frac{\omega-\mu}{\sqrt{\sigma_x^2+\sigma_{\bar{y}|x}^2}}) - \Phi(\frac{\upsilon-\mu}{\sqrt{\sigma_x^2+\sigma_{\bar{y}|x}^2}})$$

$$+BVN(\frac{\upsilon-\mu}{\sqrt{\sigma_x^2+\sigma_{\bar{y}|x}^2}};-\frac{L-\mu}{\sigma_x};-\gamma)$$

$$-BVN(\frac{\omega-\mu}{\sqrt{\sigma_x^2+\sigma_{\bar{y}|x}^2}};-\frac{L-\mu}{\sigma_x};-\gamma) \quad , \quad (19)$$

Using Eqs . (18) and (19),the customer risk *R*c can be written as

$$Rc = \beta = \Phi(\frac{\omega-\mu}{\sigma_{\bar{y}}}) - \Phi(\frac{\upsilon-\mu}{\sigma_{\bar{y}}})$$

$$+BVN(\frac{\omega-\mu}{\sigma_{\bar{y}}};-\frac{U-\mu}{\sigma_x};-\gamma) - BVN(\frac{\omega-\mu}{\sigma_{\bar{y}}};-\frac{L-\mu}{\sigma_x};-\gamma)$$

$$-BVN(\frac{\upsilon-\mu}{\sigma_{\bar{y}}};-\frac{U-\mu}{\sigma_x};-\gamma) + BVN(\frac{\upsilon-\mu}{\sigma_{\bar{y}}};-\frac{L-\mu}{\sigma_x};-\gamma)$$

$$, \quad (20)$$

IV. A Numerical Example

To demonstrate the proposed model, consider an example of indirect tensile tests of stiffness modulus, according to standard *NF EN 12697-26:2004* [10],tests on cylindrical specimens 100mm in diameter, with thickness of 53mm and 2451kg / m³ density, were carried out under the conditions listed in Table 1.

Horizontal deformation under	5 ± 2 µm
Frequency	10 Hz
Number of pulses	10
The pulse repetition period	3 ± 0.1 s
Rise Time in load	124 ± 4 ms
Poisson's ratio	0.35

Table1. Conditions of the test of the proposed model.

We obtained a Stiffness modulus $\overline{E}^*$ = 6696 MPa, with an estimated standard deviation (σ) of 382.50 MPa with $\overline{E}^* = \frac{\sum E_i^*}{10}$, *n*=10 (the average shear modulus corrected). The uncertainty has been calculated by the testing laboratory using an analytical method based on [11]. The requirements agreed upon between the customer and the supplier specified a lower specification limit, *L*, of 6000 MPa, and an upper specification limit, *U*, of 10000 MPa.

The supplier has taken ten cylindrical specimens (*n* = 10) for the control, the variability of the 10 measures is given by

$$\sigma_{y|x} = \sqrt{\frac{\sum(n_i-\bar{x})^2}{n-1}} = 296MPa \quad (21)$$

with

$$\sigma_{\bar{y}}^{\ 2} = \sigma_x^{\ 2} + \sigma_{\bar{y}|x}^{\ 2} \quad (22)$$

$$\sigma_{\bar{y}} = \sqrt{\sigma_x^{\ 2} + \sigma_{\bar{y}|x}^{\ 2}} = \sqrt{\sigma_x^{\ 2} + \sigma_{\frac{y|x}{n}}^{\ 2}} = 393.78MPa \quad (23)$$

and

$$\gamma = \frac{\text{cov}(X,\bar{Y})}{\sqrt{\text{var}(X).\text{var}(\bar{Y})}} = \frac{\sigma_x}{\sigma_{\bar{y}}} = 0.97 \quad (24)$$

Solving the mathematical model requires a lot of computing resources mainly due to the evaluation of bivariate normal probabilities. However, an approximation algorithm, developed with the free software R programming language [12], was utilized to evaluate the integrals bivariate normal.

We suppose that β_0= 0.70648 % ,The optimal solution to the example problem is found to be υ* = 6000 MPa, ω* = 9999 MPa , ε_L=1.00 MPa and
ε_U = 0.00 MPa , n = 10, with a customer risk of 0.70644 % , Guardbanding provide a way of assuring that good product would be accepted 99.23% . Numerical results are summarized in Table 2.

ε_L / ε_U	**0.00**	**0.25**	**0.50**	**0.75**	**1.00**
0.00	0.70665	0.70669	0.70675	0.70679	0.70684
0.25	0.70660	0.70664	0.70669	0.70674	0.70678
0.50	0.70654	0.70654	0.70664	0.70668	0.70673
0.75	0.70649	0.70653	0.70658	0.70663	0.70668
1.00	0.70644	0.70648	0.70653	0.70658	0.70662

Table2. Numerical results of the customer risk of the stiffness modulus.

V. Conclusion

This study has described the risk of decision error (consumer's risk) when monitoring production. A mathematical optimization model has been proposed that integrates the notion of an optimal guardbanding to minimize the impact of measurement errors and to reduce the risk of false acceptance (consumer's risk). A numerical example was provided that demonstrates the proposed optimization model.

References

[1] D. Mc Carville and D. Montgomery, Optimal guard bands for gauges in series, *Quality Engineering*, vol. 9, no. 2, pp. 167-177, (1996).

[2] B. Hutchinson, Setting guardband test limits to satisfy MIL-STD-45662A requirements, *Proc. of 1991 NCSL Workshop and Symposium*, pp. 305–309, (1991).

[3] D. Deaver, Using Guardbands to Justify TURs Less than 4:1, *Proc. of ASQC 49th Annual Quality Congress*, pp. 136-141, (1995).

[4] J. Chandra, S. Schall, The use of repeated measurements to reduce the effect of measurement errors, IIE Trans. 20, 83–87, (1988).

[5] A. Eagle, A Method for Handling Errors in Testing and Measuring,”*Industrial Quality Control*, vol. 10, no. 5, pp. 10-15, (1954).

[6] Brucker J, A note on the bivariate normal distribution, comm. Statist A8 :175-177,(1979).

[7] Fraser DAS, Streit F , A further note on the bivariate normal distribution , comm. Statist A9:1097-1099,(1980).

[8] Ahsanullah M, Some characterizations of the bivariate normal distribution , Metrika 32:215-218,(1985).

[9] S. Chen and K. Chung, Selection of the optimal precision level and target value for a production process: the lower-specification limit case, *IIE Trans*., vol. 28, no. 12, pp. 979–985, (1996).

[10] AFNOR, Méthodes d’essai pour mélange hydrocarboné à chaud : Module de Rigidité, *NF EN 12697-26*, (2004).

[11] AFNOR, Guide pour l'expression de l'incertitude de mesure, *NF ENV 13005*, (1999).

[12] P. Dalgaard, *Introductory Statistics with R*, Springer: New York, 2nd edition, (2008).