RESEARCH ARTICLE OPEN ACCESS

Modeling of the Bending Stiffness of a Bimaterial Beam by the Approximation of One-Dimensional of Laminated Theory

M. Flores-Domínguez
Institut de Mécanique et d'Ingénierie de Bordeaux, Arts et Métiers ParisTech, 33405 Talence, France

ABSTRACT
In this paper, a mathematical model representing the bending stiffness of a bimaterial beam is proposed. The classical laminated plate theory specialized to 1D is used for modeling the stiffness bending of a beam. Different material configurations (metal-polymer, metal-composite material and metal-metal) with three different ratio**s** of layer thickness were evaluated by the analytical model were proposed. Virtual experiments by finite element analysis were carried out to verify the accuracy of the proposed approach. Finite element models of each arrangement were built and the recommendations for the ASTM three-point bending test were followed in the numerical simulation. The average difference of the stiffness results calculated by the analytical model and by finite element simulation was less than 2.11%.
***Keywords* -** Bending stiffness, finite element analysis, laminated theory, multi-material.

I. INTRODUCTION

During the last decades, engineers have tried to optimize their product design by integrating increasing numbers of functions in the properties of a material. This tendency has enabled complex assemblies to be replaced by simpler structures, involving the design of new materials when none of the classical monolithic materials embodies all the required functions. Thus, in the domain of most applications, various types of multi-materials have been proposed.

A multi-material or an assembly system of materials can be defined as a combination of two or more materials in a predetermined geometry and scale [1, 2]. To define a multi-material answering a set of requirements, the designer is confronted with an infinity of potential solutions among which he has to make as objective as possible choices.

The composite laminate materials are an alternative design solution in terms of specific strength and stiffness and they offer significant freedom to the designer by allowing, the strength and stiffness optimization of a component or structure for a particular application [3].

Designing a multi-material involves the determination of all the characteristic parameters. The most used method begins by complete description of the set of requirements, the selection of the geometry of the assembly, the load type and the materials selection to the choice of the multi-materials components in order to allow a quantified evaluation of its performance [4-6].

The mechanical properties of the multi-material are a function of the choice of materials that form it and its geometrical arrangement within the structure. For laminated materials the mathematical models describe their behavior [7]. The flexural behavior of multi-material beam has been studied extensively by many investigators [8-10]. A model based on classical laminated plate theory reduced to one-dimension was developed to obtain the elastic modulus of a bimaterial [11].

The aim of this study is to obtain a simple mathematical model that describes the behavior of a multi-material beam subjected to a three-point bending. The interest is to determine the bending stiffness of the beam formed by two different materials. In this work, a one-dimensional (1D) bending model of a bimaterial structure is developed in a convenient way to obtain the bending stiffness of the bimaterial. The model is validated with a virtual experimental analysis by finite element analysis where the recommendations of the ASTM for three-point bending tests were followed.

II. LAMINATE ANALYSIS

A perfectly bonded bimaterial subject to a moment is analyzed, and is based on classical laminated plate theory (CLPT), specialized 1D. The bimaterial is of length *L*, width *b*, and thicknesses h_1 and h_2 for each material, see Fig. 1. The *x*-coordinate is the axial coordinate and the *z*-coordinate is the through-thickness coordinate, with $z = 0$ at the mid-plane $((h_1 + h_2)/2)$. A concentrated load P is applied at mid span. In this manner, the bimaterial system may be modeled using the first-order laminated theory [12], here specialized to 1D. Within the linear elastic region, the stresses (σ) are proportional to strains (ε),

$$\sigma_x(i) = E_i \varepsilon_x(x) \qquad (1)$$

Here, subscripts $i = 1, 2$ correspond to each material, and *E* is the elastic modulus for isotropic materials, or the effective modulus for composite materials.

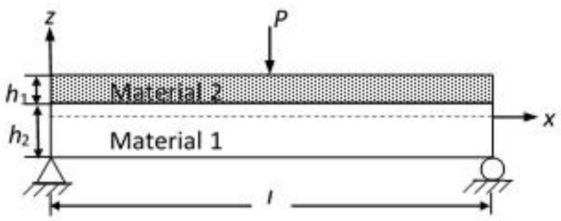

Figure 1. Schematic representation of a bimaterial system

According to the Kirchhoff hypothesis [12, 13] (see Fig. 2), the axial displacement *u* of a point at (*x, z*) may be calculated using the mid-plane axial displacement u^0 and the rotation of the cross section $\psi(x)$, this is:

$$u = u^0(x) + z\psi \qquad (2)$$

where $u^0 = u(x, z = 0)$, and ψ is the rotation of a cross section at *x*, originally plane and perpendicular to the specimen axis.

The corresponding variation of strain through the thickness is given by:

$$\varepsilon = \varepsilon_x^0 + z\kappa_x \qquad (3)$$

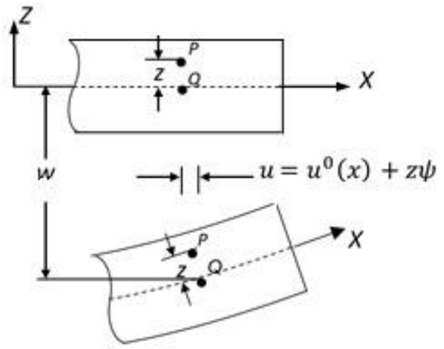

Figure 2. Deformed geometry of the section of a beam in the theory of laminated.

where ε_x^0 is the mid-plane strain, and κ_x is the mid-plane curvature given by:

$$\kappa_x = \frac{d\psi}{dx} \qquad (4)$$

In general, the force and moment resultants, Nx and Mx, are defined as [14],

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz \qquad (5a)$$

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz \qquad (5b)$$

Substitution of Equation 1 and 3 into Equation 5 yields, after integration and simplification

$$N_x = A\varepsilon^0 + B\kappa_x \qquad (6a)$$

$$M_x = B\varepsilon^0 + D\kappa_x \qquad (6b)$$

where A, B, and D are, respectively, the 1D extensional, coupling and bending stiffness, given by:

$$A = E_1 h_1 + E_2 h_2 \qquad (7a)$$

$$B = \frac{1}{2} h_1 h_2 (E_2 - E_2) \qquad (7b)$$

$$D = \frac{1}{12}[E_1(h_1^3 + 3h_1 h_2^2) + E_2(h_2^3 + 3h_1^2 h_2)] \qquad (7c)$$

It is assumed that both materials have a moderate inter-laminar shear modulus and a large (>10) length-to-thickness ratio. Thus, shear deformation is expected to be minor and can be neglected.

For the case of bending loading examined herein, the only applied load is the concentrated load P at mid span. The moment associated with the simply supported beam with a concentrated load is given as:

$$M_x = \begin{cases} \frac{P}{2b}x,\ 0 < x \le \frac{L}{2} \\ \frac{P}{2b}(L-x),\ \frac{L}{2} < x \le L \end{cases} \quad (8)$$

thus, Equation 6a with $N_x = 0$ yield

$$\varepsilon_x^0 = -\frac{B}{A}\kappa_x \quad (9a)$$

Substituting Equation 9a into Equation 6b yields the mid-plane curvature κ_x;

$$\kappa_x = \frac{M_x}{\left(D - \frac{B^2}{A}\right)} \quad (9b)$$

Since $\kappa_x = \partial^2 w(x)/\partial x^2$, the deflection of the mid-plane can be found by integrating the Equation 9b, where the conditions boundary $\partial w/\partial x = 0$ (slope of the bending) at $x = L/2$ and $w = 0$ (deflection) at $x = 0$ are evaluated for the first and second integral constants, respectively. The deflection equation is given by the following:

$$w(x) = \frac{P}{4b}\left(\frac{A}{AD - B^2}\right)\left(\frac{x^3}{3} - \frac{L^2 x}{4}\right) \quad (10)$$

The maximum deflection δ is given in $x = L/2$, thus the maximum deflection displacement is given by:

$$\delta = \frac{PL^3}{48b}\left(\frac{A}{AD - B^2}\right) = \frac{PL^3}{48b}m \quad (11)$$

where δ is the deflection measure at the center, P is the concentrated load, L is the spam between the supports and *m* is termed as the flexural rigidity (EI for a homogeneous beam) of the beam. The slope of the load-deflection curve is termed as the bending stiffness, that is:

$$K = \frac{P}{\delta} = \frac{48b}{L^3}m \quad (12)$$

Note that if $E_1 = E_2$, the maximum defection equation $\left(\delta = PL^3/4Ebh^3\right)$ of a simply supported beam with rectangular cross section subjected to concentrated load *P* in mid-span is recovered.

III. ANALYSIS OF BENDING STIFFNESS

In order to ascertain the accuracy of the obtained analytical expression of bending stiffness for the bi-material beam, a virtual experimental analysis by finite element analysis was carried out. The tests were performed using the commercial FEM software ANSYS Mechanical APDL v14.5. In this analysis, the recommendations of the ASTM for three-point bending tests [14] to characterize the stiffness of the beam were followed. The principal scheme of three-point bending test is described in Fig. 1.

As previously discussed in section 2, the bending stiffness can be determined in terms of the Young's modulus of the materials, the layers thickness, the span between supports and the width of the beam. In order to validate the Equation (12), nine different models with dimensions of 64 x 4 x 4 mm were proposed. The total thickness h of the beam was maintained constant and three different thickness ratios of the layers (h1/h2) were used: 3, 1, and 1/3. Four materials were used for the layers. The arrangements of the layers for the system were: metal-polymer, metal-composite material and metal-metal, where aluminum was used as material 2. The mechanical properties of the materials are presented in Table 1.

Table 1. Material properties used in this study.

Material	Property	
	E [GPa]	ν
Aluminum (Al)	69	0.33
Steel (St)	200	0.3
Polymer (PC)	2.38	0.36
CFRP	139	0.21

3.1 Finite Element Analysis

The 3D models of the different bimaterial beams proposed were constructed. The models consist of two laminates perfectly bonded at the interface. The models were discretized with 3-D 20-node solid element for both materials. These elements have quadratic displacement behavior and they have three degrees of freedom per node, translated into the nodal x, y and z directions [15]. For each material, the nonlinear behavior was considered and isotropic hardening rule for multi-point material model was used.

To specify the three-point bending test, the nodes at the end left bottom of the beam were constrained in the z translational displacement and the nodes at the end right bottom of the beam were fully constrained. The load was applied in negative z direction of the beam at half the length along the entire width of the beam and it was applied in steps from zero to the von Mises stress of any of the two materials was equal to the ultimate stress of its material.

Example of the discretized models is shown in Fig. 3. The model for material configurations CFRP-Aluminum with 1/3 thickness ratio is presented. The mesh was formed by 5825 nodes and 1024 elements. Detail of the material thickness 2 can be observed.

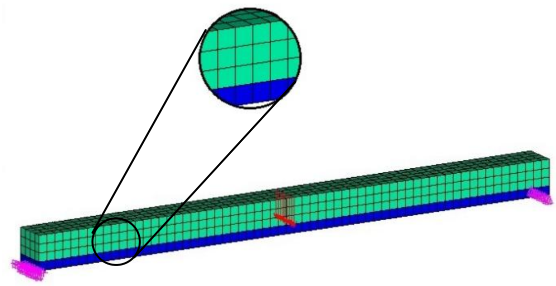

Figure 3. FE model of three-point bending test.

IV. RESULTS AND DISCUSSION

The evaluation of the stiffness of the bi-material beam was conducted. The bending stiffness K of the beam was calculated using the analytical model and was compared with the results of numerical simulation.

In the finite element analysis, the transverse deflection uz was evaluated from the successful execution of the ANSYS software after conducting several convergence tests. From the ultimate load and the maximum displacement in z direction that each model of the beam presents, the bending stiffness was calculated using the equation $K= P/\delta$. Fig. 4 presents the load-deflection curves obtained by simulation of the different proposed models.

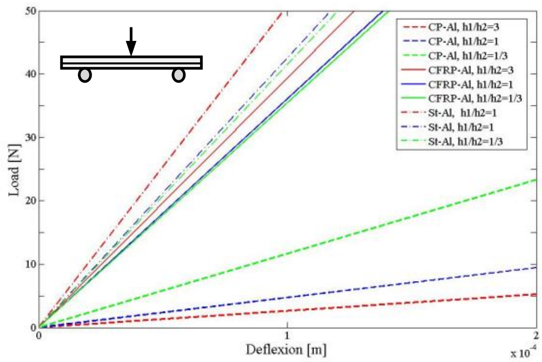

Figure 4. Load-deflection curves, obtained by simulation of three-point bending test. The slopes are the stiffness of different models.

Example of the contour of displacement and stress obtained of the finite element analysis are shown in Fig. 5 and 6 respectively. In Fig. 5, the transverse displacement uz of the material configuration CFRP-Al beam with h1/h2=1 is depicted. The maximum displacement is observed at mid spam. The Von Mises stress of the same beam is observed in Fig. 6.

The obtained values by the laminated plate theory specialized for 1D and by finite element simulation are listed in Table 2. The stiffness determined by analytical model and numerical simulation, are compared for each of the proposed models.

In both analyses were observed, that for PC-Al arrangement, the stiffness increased with decreasing of the ratio thickness. On the other hand, for CFRP-Al and St-Al arrangements, the stiffness decreased with decreasing of the ratio thickness. That is, the stiffness bending increases with increasing the thickness of the material with higher Young's modulus.

Table 2 shows that the analytical results are close to finite element simulation results. The values of bending stiffness obtained by the analytical model are higher than the values obtained by finite element simulation. The differences in the stiffness calculated by analytical model and simulation were less than 3.5% and the average difference was less than 2.11%. The results obtained by the analytical model yielded a good agreement with the finite element results obtained.

The proposed analytical model equation illustrates the interaction between different variables. In this study, the combination of the different Young's modulus of the materials and the thickness ratio were observed. Accordingly, two response surface graphs were generated.

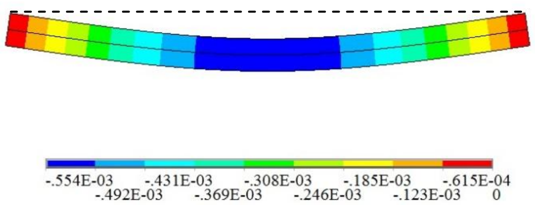

Figure 5. Deflection of the CFRP-Aluminum beam with thickness ratio 1.

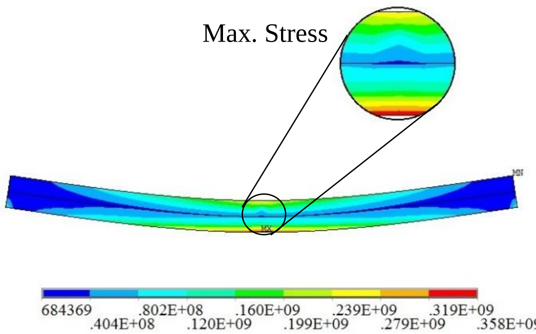

Figure 6. Stress presented in the CFRP-Aluminum beam with thickness ratio 1.

Table 2. Bending Stiffness calculated by analytical model and finite element simulation

Material configurations	Ratio h1/h2	K [10^3 N/m]		% Δ
		Analytical	Simulation	
Al-CP	3	27.1	26.2	3.27
Al-CP	1	48.3	47.3	2.22
Al-CP	1/3	120.8	116.6	3.48
Al-CFRP	3	399.4	393.7	1.43
Al-CFRP	1	367.2	361	1.67
Al-CFRP	1/3	361.3	354.6	1.86
Al-St	3	515.1	507.6	1.45
Al-St	1	431.9	425.6	1.48
Al-St	1/3	423.9	414.9	2.12

The behavior of the analytical model obtained by the laminated plate theory specialized to 1D as function of the Young's modulus of the laminated materials, is presented in Fig. 7. In this figure, the response surface reveals that an increase in Young's modulus of any layers cause an increase in the bending stiffness and localized the optimum values of each Young's modulus for maximum response. The representation of the analytical model of the bending stiffness considering the total thickness of the beam and the thickness ratio of the layers can be observed in Fig. 8. This figure shown that the total thickness variable has a greater effect on the response to the increasing the stiffness of beam to a greater extent. This fact can be explained by the greater amount of beam material, with the same ratio of thickness.

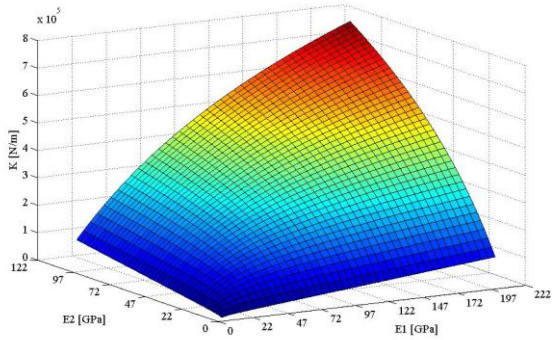

Figure 7. Response surface described by the model analytical, which represents bending stiffness as a function of the Young's modulus of the laminated materials.

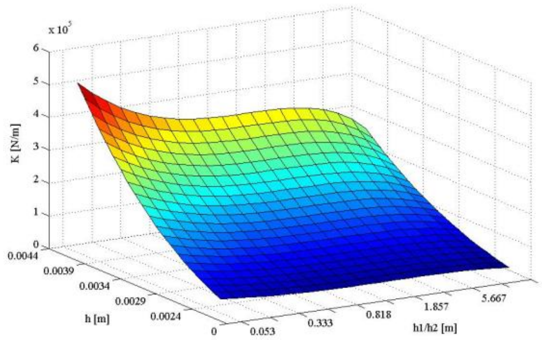

Figure 8. Response surface for the bending stiffness as a function of the total thickness of the beam and the thickness ratio of the layers.

V. Conclusions

A simplified analytical model to characterize the bending stiffness of a bimaterial beam was presented. Analysis based on classical laminated plate theory (CLPT) specialized to 1D was carried out to obtain the mathematical model of the stiffness. Models of different material configurations and different thickness ratio of the layer were proposed to calculate the bending stiffness. These models were analyzed using the mathematical model and by experiments performed by finite element simulation. The flexural response of the bimaterial beam by numerical simulation was studied for three-point bending configuration. The differences in the stiffness calculated by analytical model and simulation were less than 3.5% and the average difference was less than 2.11%. It was found that the analytical solution provided good agreement with the experimental results. This mathematical model can be used with different configurations material and layer thicknesses. The analysis could be readily extended to multilayers.

Acknowledgements

Flores wishes to express their gratitude to the Council of Science and Technology of Mexico State (COMECYT), Mexico for their financial support.

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