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Mathematical Modeling of Bingham Plastic Model of Blood Flow Through Stenotic Vessel

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Abstract
The aim of the present paper is to study the axially symmetric, laminar, steady, one-dimensional flow of blood through narrow stenotic vessel. Blood is considered as Bingham plastic fluid. The analytical results such as pressure drop, resistance to flow and wall shear stress have been obtained. Effect of yield stress and shape of stenosis on resistance to flow and wall shear stress have been discussed through tables and graphically. It has been shown that resistance to flow and the wall shear stress increase with the size of stenosis but these increase are, however, smaller due to non-Newtonian behaviour of the blood.
Key words : Laminar flow, pressure drop, resistance to flow, wall shear stress, stenosis.

I. INTRODUCTION

The theoretical and experimental investigations of blood flow through arteries are of considerable importance in many cardiovascular diseases viz. atherosclerosis, atherogenesis, atheroma etc. are closely associated with the flow conditions in the blood vessels. The normal flow of blood is disturbed due to some abnormal growths like stenosis in the lumen of the artery. Due to localised accumulation of material within the intima (i.e. the inner surface of arteries), the deposits sometimes turn into atherosclerotic plaques, greatly reducing the arterial diameter. It has been seen through clinical examinations that such a condition can lead to hemorrhage and local thrombosis. The actual reason of abnormal growth in an artery is not completely clear but its effect over the cardiovascular system has been determined by studying the flow characteristics of blood in the stenosis area. The partial occlusion of a coronary artery can lead to angina pectoris and there will be increased risk of myocardial infraction. This type of occlusion in the vessel carrying blood to the limbs can cause severe pain and loss of the function. So many investigations are performed on the prevention and cure of atherosclerosis. The results of these investigations promised better understanding of the nature of this type of disease. An attempt towards a systematic study of the flow around a stenosis seems to have been started by Young [1].

The effects of flow separation were examined by Forrester and Young [2]. Young and Tsai [3] performed an experiment on the models of arterial stenosis by considering the steady flow of blood and reported that the hydrodynamic factors play a significant role in the development and progression of the arterial stenosis. Several researchers (Young [4], Morgan and Young [5], Chakravarty and Chowdhury [6], MacDonald [7], Ponalagusamy [8], Sanyal and Maji [9], Gupta and Gupta [10], Misra and Chakravarty [11], Haldar [12], Farzan Ghalichi et al. [13], Verma and Anuj Srivastava [14], Verma [15]) have studied the flow characteristics of blood in an artery with mild stenosis while the fluid representing blood has been considered to be Newtonian. However, it may be noted that blood, being a suspension of red cells in plasma, behaves like a non-Newtonian fluid at low shear rates in smaller diameter tubes (Huckaba and Hahn [16], Charm and Kurland [17], Whitmore [18]. Many investigators carried out a good number of studies on various aspects associated with blood flow through small stenosed vessels by proposing different non-Newtonian model (Shukla et al. [19], Shukla et al. [20], Chaturani and Ponnalagar Samy [21], Sapna Ratan Shah [22], Saleh and Khan [23], Sanjeev Kumar and Archana Dixit [24], Bijendra Singh et al. [25]). Shukla et al. [19] examined the effects of stenosis on the resistance to flow and wall shear stress in an artery by assuming blood as a power-law and Casson's fluid and concluded that the resistance to flow and wall shear stress increases as the size of the stenosis increases but this increase is very small due to non-Newtonian behaviour of the blood.

With the above discussion in mind, a theoretical analysis of the flow characteristics of blood in an arterial segment having a stenosis is presented in this paper. It is assumed that the wall of vessel is regid and containing non-Newtonian Bingham plastic viscous incompressible fluid (blood). The results of the paper which have been computed on the basis of the present analysis are expected as a good agreement with other studies.

II. MATHEMATICAL FORMULATION

Let us consider steady flow of incompressible, one-dimensional, axially symmetric, fully developed laminar flow of blood through narrow vessel (circular tube or artery) with stenosis as shown in Fig.1. Here cylindrical co-ordinate system is used. The blood flow is assumed to be characterised by Bingham plastic fluid model. The radius of the tube is given by

$$\frac{R_{(z)}}{R_0} = 1 - \frac{\delta_h}{2R_0}\left\{1 + Cos\frac{2\pi}{L_0}\left(z - d - \frac{L_0}{2}\right)\right\}; d \le z \le L_0 + d$$

$$= 1 \qquad ; \text{otherwise} \qquad (1)$$

where L_0 is the length of stenosis, *d* indicates its location and δ_h is the maximum height of the stenosis $R_{(z)}$ is the radius of the artery in the stenotic region, R_0 is the constant radius $(\delta_h \ll R_0, Fig.1)$

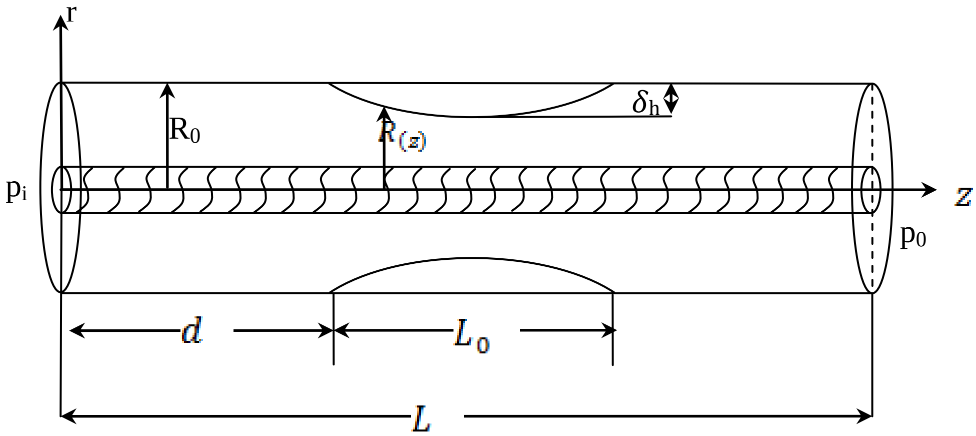

Fig.1: Geometry of stenosed vessel

The approximate equation of motion governing the flow field in the tube is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) = -\frac{dp}{dz}, \qquad (2)$$

where τ_{rz}, for Bingham plastic fluid, is given by

$$-\frac{dw}{dr} = f(\tau) = \left[\frac{\tau_{rz}-\tau_0}{\mu}\right] \qquad ; \qquad \tau_{rz} \ge \tau_0$$

$$\frac{dw}{dr} = f(\tau) = 0 \qquad : \qquad \tau_{rz} \le \tau_0 \qquad (3)$$

where *z* and *r* are axial and radial coordinates, *w is* the axial velocity, τ_{rz} is the shear stress, τ_0 is the yield stress, μ is the viscosity of the blood, *p* is the fluid pressure. The region $\tau_{rz} \le \tau_0$ implies a plug flow in which velocity gradient vanishes.

The appropriate boundary conditions are

$$w = 0 \qquad \text{on } r = R(z), \text{ no slip at the wall} \qquad (4)$$

$$\frac{dw}{dr} = 0 \qquad \text{on } r = 0, \text{ symmetry about the axis} \qquad (5)$$

$$\tau_{rz} \qquad \text{is finite at } r = 0 \qquad (6)$$

Following Whitemore [18], Bird et al [26] and Govier and Aziz [27], the volumetric flow rate, *Q*, can be written in the form of a Robinowitsch equation as :

$$Q = \frac{\pi R^3}{\tau_R^3}\int_0^{\tau_R} \tau_{rz}^2 f(\tau) d\tau, \qquad (7)$$

where $$\tau_R = -\frac{R}{2}\frac{dp}{dz} \qquad ; \qquad (8)$$

τ_R is the shear stress at the wall and $\frac{dp}{dz}$ is the pressure gradient.

III. ANALYSIS

Substituting the value of $f(\tau)$ from equation (3) into equation (7) and then integrating, we obtain

$$Q = \frac{\pi R^3}{\mu} \left[\frac{\tau_R}{4} - \frac{\tau_0}{3}\right] \tag{9}$$

Using equation (8) in equation (9), we have pressure gradient as

$$-\frac{dp}{dz} = \frac{2}{R}\left[\frac{4}{3}\tau_0 + 4\mu\frac{Q}{\pi R^3}\right] \tag{10}$$

where Q is independent of z.

Integrating equation (10) and using the conditions $p = p_i$ at $z = 0$ and $p = p_0$ at $z = L$, the length of the vessel, we get

$$p_i - p_0 = \frac{8}{3}\frac{\tau_0}{R_0}\int_0^L \frac{dz}{(R/R_0)} + \frac{8\mu Q}{\pi R_0^4}\int_0^L \frac{dz}{(R/R_0)^4}, \tag{11}$$

where R/R_0 is given by equation (1).

The resistance to flow, λ, is defined as

$$\lambda = \frac{p_i - p_0}{Q} \tag{12}$$

Using equations (1) and (11), in equation (12), we have

$$\lambda = \frac{8}{3}\frac{\tau_0}{QR_0}\left[\int_0^d dz + \int_d^{d+L_0}\frac{dz}{(R/R_0)} + \int_{d+L_0}^L dz\right] + \frac{8\mu}{\pi R_0^4}\left[\int_0^d dz + \int_d^{d+L_0}\frac{dz}{(R/R_0)^4} + \int_{d+L_0}^L dz\right] \tag{13}$$

For normal vessel $(\delta_h = 0 \; or \; R = R_0)$, the resistance to flow, λ_N is obtained from equation (13) as

$$\lambda_N = \frac{8}{3}\frac{\tau_0 L}{QR_0} + \frac{8\mu L}{\pi R_0^4} \tag{14}$$

The ratio λ/λ_N is obtained from equation (13) and (14), by using equation (1) as

$$\frac{\lambda}{\lambda_N} = \frac{1}{f_1+f_2}\left[f_1\left\{1 - \frac{L_0}{L} + \frac{1}{2\pi}\frac{L_0}{L}\int_0^{2\pi}\frac{d\phi}{(a+b\cos\phi)}\right\} = +f_2\left\{1 - \frac{L_0}{L} + \frac{1}{2\pi}\frac{L_0}{L}\int_0^{2\pi}\frac{d\phi}{(a+b\cos\phi)^4}\right\}\right], \tag{15}$$

where

$$f_1 = \frac{8}{3}\frac{\tau_0}{QR_0}, \qquad f_2 = \frac{8\mu}{\pi R_0^4} \tag{16}$$

$$a = 1 - \frac{\delta_h}{2R_0}, \qquad b = \frac{\delta_h}{2R_0}, \tag{17}$$

$$\phi = \pi - \frac{2\pi}{L_0}\left(z - d - \frac{L_0}{L}\right). \tag{18}$$

The value of second integral of equation (15) can be calculated by the method of calculus of residues and solution of (15) is obtained as

$$\frac{\lambda}{\lambda_N} = 1 - \frac{L_0}{L} + \frac{L_0}{L}\left[\frac{f_1 g_1 + f_2 g_2}{f_1+f_2}\right], \tag{19}$$

where

$$g_1 = \frac{1}{\sqrt{a^2-b^2}} \tag{20}$$

$$g_2 = \frac{1}{(a^2-b^2)^2}(1 - 12s + 30s^2 - 20s^3), \tag{21}$$

$$s = \frac{-a+\sqrt{a^2-b^2}}{2\sqrt{a^2-b^2}} \quad . \tag{22}$$

When $\tau_0 = 0$ in equation (19), the resistance to flow ratio is same as obtained by Young [1] for Newtonian fluid.

From equation (8) and (10), the wall shear stress ratio τ_R/τ_N can be obtained as

$$\frac{\tau_R}{\tau_N} = \frac{1}{f_1+f_2}\left[f_1 + \frac{f_2}{(R/R_0)^3}\right] , \tag{23}$$

where τ_N is the shear stress at the wall for normal vessel $(\delta_h = 0)$, which is given by

$$\tau_N = \frac{4}{3}\tau_0 + 4\mu\frac{Q}{\pi R_0^3} \quad . \tag{24}$$

When $\tau_0 = 0$ in equation (23), τ_R/τ_w is obtained for Newtonian fluid.

The wall shear stress at the throat of stenosis is obtained from equation (23) as

$$\frac{\tau_s}{\tau_N} = \frac{1}{f_1+f_2}\left[f_1 + \frac{f_2}{\left(1-\frac{\delta_h}{R_0}\right)^3}\right]. \tag{25}$$

For Newtonian case $(\tau_0 = 0)\tau_s/\tau_N$ can be written as

$$\frac{\tau_s}{\tau_N} = \frac{1}{\left(1-\frac{\delta_h}{R_0}\right)^3} \quad . \tag{26}$$

IV. DISCUSSION

For the purpose of computation we have made use of the following values:

$Q = 2 \times 10^{-4}\ cm^3/s$, $\tau_0 = 0.00 - 0.05\ dyne/cm^2$,

$\mu = 4.03 \times 10^{-2}\ dyne.s/cm^2$, $R_0 = 35 \times 10^{-4}\ cm$,

$\frac{L_0}{L} = 0.1 - 1.0$, $\frac{\delta_h}{R_0} = 0.0 - 0.5$

Numerical results for $\lambda/\lambda_N, \tau_R/\tau_N$ and τ_s/τ_N are obtained through the use of the above mentioned data are exhibited in figure 1, 2 and table 2-4.

The dimensionless resistance to flow, λ/λ_N, vs stenosis height δ_h/R_0, is plotted for different values of L_0/L in Fig.2. It may be noted from Fig.2 that the resistance to flow increases as the size of the stenosis increases. This increase is large for severe case. Further, for $L_0/L = 1$ and $\tau_0 = 0.03$ the resistance to Flow λ/λ_N is increased over that for uniform diameter vessel by 24.239%, 62.9%, 123.088%, 232.962% and 457.582% for $\delta_h/R_0 = 0.1, 0.2, 0.3, 0.4$ and 0.5 respectively.

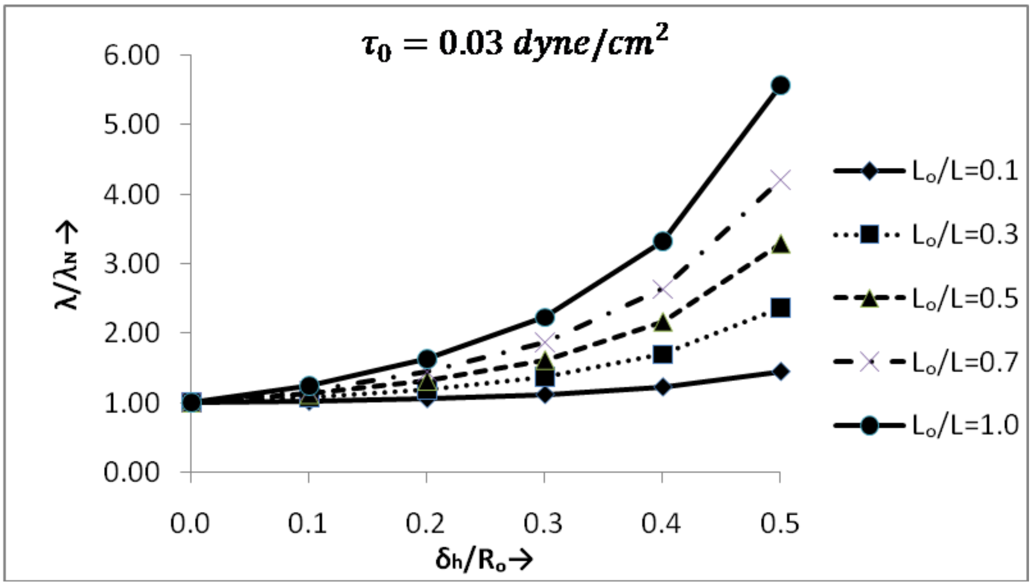

Fig.2: Variation of λ/λ_N with δ_h/R_0 for different L_0/L.

From tables 1 and 2 we see that λ/λ_N decreases with yield stress for fixed stenosis height and L_0/L. This decrease is very small due to non-Newtonian behaviour of blood. It is observed that the values of λ/λ_N for Newtonian case are greater than that of non-Newtonian case.

Table-1

Variation of λ/λ_N with δ_h/R_0 for different $\tau_0, (L_0/L = 1.0)$

$\frac{\delta_h}{R_0} \rightarrow$ / τ_0 dyne/cm^2 ↓	0.0	0.1	0.2	0.3	0.4	0.5
0.00	1.00000	1.24242	1.62909	2.23105	3.32996	5.57670
0.01	1.00000	1.24240	1.62906	2.23099	3.32985	5.57647
0.03	1.00000	1.24239	1.62900	2.23088	3.32962	5.57582
0.05	1.00000	1.24237	1.62895	2.23076	3.32939	5.57554

Table-2

Variation of λ/λ_N with L_0/L for different $\tau_0, (\delta_h/R_0 = 0.3)$

$\frac{L_0}{L} \rightarrow$ / τ_0 dyne/cm^2 ↓	0.1	0.3	0.5	0.7	1.0
0.00	1.12311	1.36932	1.61553	1.8674	2.23105
0.01	1.12310	1.36930	1.61550	1.86169	2.23099
0.03	1.12309	1.36926	1.61544	1.86162	2.23088
0.05	1.12308	1.36923	1.61538	1.86153	2.23076

The expression for the wall shear stress τ_R/τ_N is plotted in Fig. 3 with axial distance for different δ_h/R_0, which shows that τ_R/τ_N increases with z/L_0 for fixed stenosis height upto throat and then decreases. The variation of τ_R/τ_N with axial distance z/L_0 for different yield stress τ_0 is given in table 3. It is observed that at fixed z/L_0, τ_R/τ_N decreases as τ_0 increases i.e. the values for non-Newtonian are smaller than that Newtonian values. The variation of wall shear stress at the throat of stenosis with δ_h/R_0 for different τ_0 is shown in table 4. It is clear that τ_s/τ_N increases with δ_h/R_0 for fixed τ_0 and which is very large for severe case.

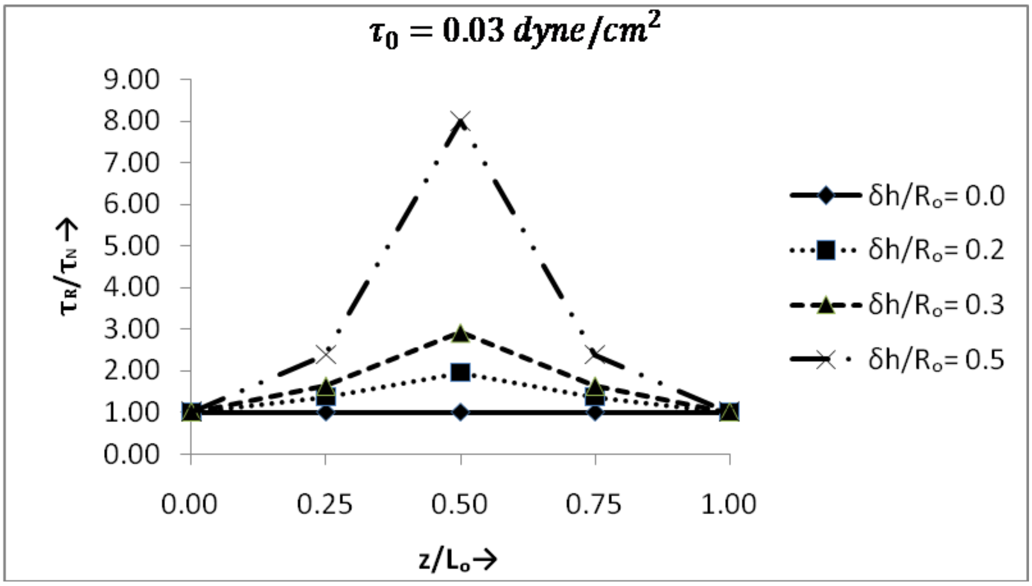

Fig.3: Variation of τ_R/λ_N with z/L_0 for different δ_h/R_0 .

Table-3

Variation of τ_R/τ_N with z/L_0 for different $\tau_0, (\delta_h/R_0 = 0.3)$

$\frac{z}{L_0}$ → / τ_0 dyne/cm^2 ↓	0.00	0.25	0.50	0.75	1.00
0.00	1.00000	1.62833	2.91545	1.62833	1.00000
0.01	1.00000	1.62830	2.91535	1.62830	1.00000
0.03	1.00000	1.62823	2.91513	1.62823	1.00000
0.05	1.00000	1.62816	2.91492	1.62816	1.00000

Table-4

Variation of τ_s/τ_N with δ_h/R_0 for different τ_0.

$\frac{\delta_h}{R_0}$ → / τ_0 dyne/cm^2 ↓	0.0	0.1	0.2	0.3	0.4	0.5
0.00	1.00000	1.37174	1.95312	2.91545	4.62963	8.00000
0.01	1.00000	1.37172	1.95307	2.91534	4.62943	7.99961
0.03	1.00000	1.37168	1.95297	2.91513	4.62903	7.99883
0.05	1.00000	1.37164	1.95286	2.91492	4.62862	7.99806

Finally, it may be concluded that in the present analysis, the effect of the stenosis is more on Newtonian fluid than on non-Newtonian fluid. The effect of yield stress is very small on the flow.

V. ACKNOWLEDGEMENT

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