Triple Domination Number and Its Connectivity of Complete Graphs

A. Nellai Murugan\textsuperscript{1} G. Victor Emmanuel\textsuperscript{2}

Assoc. Prof. of Mathematics, V. O. Chidambaran College, Thoothukudi-628 008, Tamilnadu, India\textsuperscript{1}

Asst. Prof. of Mathematics, St. Mother Theresa Engineering College, Thoothukudi-628 102, Tamilnadu, India\textsuperscript{2}

\section*{ABSTRACT}
In a graph $G$, a vertex dominates itself and its neighbours. A subset $S$ of $V$ is called a dominating set in $G$ if every vertex in $V$ is dominated by at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A set $S \subseteq V$ is called a Triple dominating set of a graph $G$ if every vertex in $V$ is dominated by at least three vertices in $S$. The minimum cardinality of a triple dominating set is called Triple domination number of $G$ and is denoted by $T\gamma(G)$. The connectivity $\kappa(G)$ of a connected graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper we find an upper bound for the sum of the Triple domination number and connectivity of a graph and characterize the corresponding extremal graphs.

\textbf{Mathematics Subject Classification:} 05C69

\textbf{Key Words:} Domination, Domination number, Triple domination, Triple domination number and connectivity.

\section{INTRODUCTION}
The concept of domination in graphs evolved from a chess board problem known as the Queen problem- to find the minimum number of queens needed on an $8 \times 8$ chess board such that each square is either occupied or attacked by a queen. C. Berge\textsuperscript{[12]} in 1958 and 1962 and O. Ore\textsuperscript{[11]} in 1962 started the formal study on the theory of dominating sets. Thereafter several studies have been dedicated in obtaining variations of the concept. The authors in\textsuperscript{[2]} listed over 1200 papers related to domination in graphs in over 75 variation.

The graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. The degree of any vertex $u$ in $G$ is the number of edges incident with $u$ is denoted by $d(u)$. The minimum and maximum degree of a graph $G$ is denoted by $\delta(G)$ and $\Delta(G)$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak\cite{1} and Haynes et.al\cite{2,3}.

Let $v \in V$. The open neighbourhood and closed neighbourhood of $v$ are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$ then $N(S) = \bigcup_{v \in S} N(v)$ for all $v \in S$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbourhood of $u$ with respect to $S$ is defined by $p_n[u, S] = \{v : N[v] \cap S = \{u\}\}$. We denote a cycle on $n$ vertices by $C_n$, a path on $n$ vertices by $P_n$ and a complete graph on $n$ vertices by $K_n$. A bipartite graph is a graph whose vertex set can be divided into two disjoint sets $V_1$ and $V_2$ such that every edge has one end in $V_1$ and another in $V_2$. A complete bipartite graph is a bipartite graph where every vertex of $V_1$ is adjacent to every vertex in $V_2$. The complete bipartite graph with partitions of order $|v_1| = m$, and $|v_2| = n$ is denoted by $K_{m,n}$. A wheel graph, denoted by $W_n$ is a graph with $n$ vertices formed by connecting a single vertex to all vertices of $C_{n-1}$. $H(m_1, m_2, \ldots, m_n)$ denotes the graph obtained from the graph $H$ by pasting $m_i$ edges to the vertex $v_i \in V(H)$, $1 \leq i \leq n$. $H(P_{m_1}, P_{m_2}, \ldots, P_{m_n})$ is the graph obtained from the graph $H$ by attaching the end vertex of $P_{m_i}$ to the vertex $v_i$ in $H$, $1 \leq i \leq n$. Bistar $B(r,s)$ is a graph obtained from $K_{2r}$ and $K_{1,s}$ by joining its centre vertices by an edge.

In a graph $G$, a vertex dominates itself and its neighbours. A subset $S$ of $V$ is called a dominating set in $G$ if every vertex in $V$ is dominated by at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. Harary and Haynes\cite{4} introduced the concept of double domination in graphs. A set $S \subseteq V$ is called a double dominating set of a graph $G$ if every vertex in $V$ is dominated by at least two vertices in $S$. The minimum cardinality of double dominating set is called double domination number of $G$ and is denoted by $dd(G)$. A vertex cut, or separating set of a connected graph $G$ is a set of vertices whose removal results in a disconnected graph. The connectivity or vertex connectivity of a graph $G$ denoted by $\kappa(G)$ . The Connectivity $\kappa(G)$ of a connected graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. A matching $M$ is a
subset of edges so that every vertex has degree at most one in M.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J.Paulraj Joseph and S. Arumugam[5] proved that \( \gamma(G) + \kappa(G) \leq n \) and characterized the corresponding extremal graphs. In this paper, we obtained an upper bound for the sum of the Triple domination number and connectivity of a graph characterized the corresponding extremal graphs. We use the following theorems.

**Theorem 1.1.** [2] For any graph G, \( dd(G) \leq n \)

**Theorem 1.2.** [1] For a graph G, \( \kappa(G) \leq \delta(G) \)

### II. MAIN RESULTS

**Definition 2.1**

A set \( S \subseteq V \) is called a **Triple dominating set** of a graph G. If every vertex in V is dominated by at least three vertices in S. The minimum cardinality of Triple dominating set is called **Triple domination number** of G and is denoted by \( T\gamma(G) \). The connectivity \( \kappa(G) \) of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Choose \( S=\{v_1,v_2,v_3\}\subseteq V(G) \), if \( N[S]=V(G) \). A dominating set obtained in the way given above is called a Triple dominating set.

**Example 2.2.** For the graph \( K_n \), \( \gamma(G) = n \) and \( \kappa(G) = n \). We have \( \gamma(G) + \kappa(G) = 2n \).

**Example 2.3.** For the graph \( K_n \), \( \gamma(G) = n \) and \( \kappa(G) = n \). We have \( \gamma(G) + \kappa(G) = 2n \).

**Example 2.4.** For the graph \( K_n \), \( \gamma(G) = n \) and \( \kappa(G) = n \). We have \( \gamma(G) + \kappa(G) = 2n \).

**Lemma 2.6.** For any graph \( K_n \). We have \( T\gamma(G) + \kappa(G) \leq 2n - 1 \) when \( n=1 \) and \( T\gamma(G) + \kappa(G) \leq 2n - 2 \) when \( n=2 \).

**Result 2.7.** [14] For any graph \( K_n \). We have \( \gamma(G) \leq \gamma(G) \leq T\gamma(G) \). Where \( \gamma(G) \) is the degree equitable domination number of G.

**Lemma 2.8.** For any graph G, \( T\gamma(G) \leq n \).

**Lemma 2.9.** For any connected graph G, \( \kappa(G) = n - 1 \) if G is isomorphic to \( K_n \).

**Result 2.10.** For any connected graph G, \( T\gamma(G) = 3 \) if G is isomorphic to \( K_n \), \( n \geq 3 \).

**Result 2.11.** For the graph \( K_{n,n} \) where \( m=n \). There exists a Triple dominating set with matching M.

**Theorem 2.12.** Let \( G_1 \) and \( G_2 \) be any two graphs of Triple dominating sets then \( G_1 + G_2 \) is a graph of Triple dominating set of \( G_1 \) or \( G_2 \).

**Proof:** Let \( G_1 \) and \( G_2 \) be any two graphs having triple dominating sets. By taking sum of \( G_1 \) and \( G_2 \), we have every vertex in \( G_1 \) is adjacent to every vertex in \( G_2 \). Therefore by the definition of triple dominating set, we have By choosing S is the Triple dominating set of \( G_1 \) or \( G_2 \) and \( N[S]=V(G_1+G_2) \). Hence S is the Triple dominating set of \( G_1 \) or \( G_2 \).

**Theorem 2.13.** Every complete graph \( K_n \) has a Triple dominating set if \( n \geq 3 \).

**Proof:** Given the graph G is complete when \( n \geq 3 \). Choose \( S=\{v_1,v_2,v_3\}\subseteq V(G) \), if \( N[S]=V(G) \). A dominating set obtained is a \( T\gamma(G) = 3 \).

**Theorem 2.14.** For any connected graph G, \( T\gamma(G) + \kappa(G) = 2n - 1 \) if and only if G is isomorphic to \( K_4 \) or \( C_4 \).

**Proof:**

**Case 1.** \( T\gamma(G) + \kappa(G) = 2n - 1 \) if and only if G is isomorphic to \( K_4 \).

**Case 2.** Suppose \( T\gamma(G) + \kappa(G) = n \) and \( \kappa(G) = n \) then \( \delta(G) \leq n \). Hence \( T\gamma(G) = n - 1 \) and \( \kappa(G) = n \) is not possible.

**Theorem 2.15.** For any connected graph G, \( T\gamma(G) + \kappa(G) = 2n - 2 \) if and only if G is isomorphic to \( C_4 \).

**Proof:**

**Case 1.** \( T\gamma(G) + \kappa(G) = n - 1 \) and \( \kappa(G) = n - 1 \) and G is a complete graph on n vertices. Since \( T\gamma(G) = n \) we have \( n = 3 \). Hence G is isomorphic to \( K_4 \) or \( C_4 \).

**Case 2.** \( T\gamma(G) + \kappa(G) = 2n - 2 \) when \( n = 3 \) then there are two cases to be considered.

(i) \( T\gamma(G) = n - 1 \) and \( \kappa(G) = n - 1 \) (ii) \( T\gamma(G) = n \) and \( \kappa(G) = n - 2 \).

**Proof:**

**Case 1.** \( T\gamma(G) = n - 1 \) and \( \kappa(G) = n - 1 \). Then G is a complete graph on n vertices. Since \( T\gamma(G) = n \) we have \( n = 3 \). Hence G is isomorphic to \( K_4 \).

**Case 2.** \( T\gamma(G) = n \) and \( \kappa(G) = n - 2 \). Then \( n - 2 \leq \delta(G) \). If \( \delta(G) = n - 1 \), then G is a complete graph, which is a contradiction. Hence \( \delta(G) = n - 2 \). Then G is isomorphic to \( K_{n-1} \) where M is a matching in \( K_n \). Then \( T\gamma(G) = 3 \) or 4. If \( T\gamma(G) = 3 \) then \( n = 3 \). Which is a contradiction to \( \kappa(G) \geq n \). Thus \( T\gamma(G) = 4 \). Then \( n = 4 \) and hence G is isomorphic to \( K_4 \).

Choose \( G_1 = \alpha \) and \( G_2 = \beta \) with \( |\alpha|=1 \) or 2 respectively.

![fig:1a](image1.png)

![fig:1b](image2.png)
Case 1.

$Tγ(G) = n-2$ and $κ(G) = n-1$. Then $G$ is a complete graph on $n$ vertices. Since $Tγ(G) = 3$ and $δ(G) = n$ is not possible, we have $n = 5$. Hence $G$ is isomorphic to $K_5$.

Case 2.

$Tγ(G) = n-1$ and $κ(G) = n-2$. Then $n - 2 ≤ δ(G)$. If $δ = n - 1$ then $G$ is a complete graph, which gives a contradiction to $κ(G) = n - 2$. If $δ(G) = n - 2$, then $G$ is isomorphic to $K_n - M$ where $M$ is a matching in $K_n$, and $Tγ(G) = 3$ or $4$. If $Tγ(G) = 3$ then $n = 4$. Then $G$ is either $C_4$ or $K_4 - e$. But $Tγ(G) = 4$ implies $G$ is a dominating set of $K_4$ with $|M| = 1$. Hence $G$ is isomorphic to $K_4$ by fig:2a.

If $Tγ(G) = 4$ then $n = 5$ and hence $G$ is isomorphic to $K_5 - M$ where $M$ is a matching on $K_5$. Fig:2b.

Case 3.

$Tγ(G) = n$ and $κ(G) = n - 3$. Then $n - 3 ≤ δ(G)$. If $δ = n - 1$ then $G$ is a complete graph, which gives a contradiction to $κ(G) = n - 3$. If $δ(G) = n - 2$, then $G$ is isomorphic to $K_n - M$ where $M$ is a matching in $K_n$, and $Tγ(G) = 3$ or $4$. If $Tγ(G) = 3$ then $n = 4$ or $3$. Since $n = 4$ is impossible, we have $n = 4$. Then $G$ is isomorphic to $K_4 - e$ or $C_4$. For these two graphs, $κ(G) ≢ n - 3$ which is a contradiction. Hence $δ = n - 3$.

Let $X$ be the vertex cut of $G$ with $|X| = n - 3$ and let $V - X = \{x_1, x_2, x_3\}$, $X = \{v_1, v_2, ..., v_{n-3}\}$.

Sub Case 3.1. $(V - X) = K_3$

Then every vertex of $V - X$ is adjacent to all the vertices of $X$. Then $\{(V - X) \cup \{v_i\}\} - M$ is a Triple dominating set of $G$ where $M$ is a matching in $K_n$ and hence $Tγ(G) ≤ 4$. This gives $n = 4$. For this graph $κ(G) = 1$ which is a contradiction to $κ(G) = n - 3$. Fig:3a.

Sub Case 3.2. $(V - X) = K_1 \cup K_2$

Let $x_1x_2 \in E(G)$, Then $x_1$ is adjacent to all the vertices in $X$ and $x_1, x_2$ are not adjacent to at most one vertex in $X$. If $v_1 \notin N(x_1)$ or $N(x_2)$ then $\{(V - X) \cup \{v_i\}\}$ is a Triple dominating set of $G$ and hence $Tγ(G) ≤ 4$. This gives $n = 4$. For this graph $κ(G) = 1$ which is a contradiction to $κ(G) = n - 3$. Fig:3b.

Case 3. $Tγ(G) = n - 1$ and $κ(G) = n - 3$. Then $n - 3 ≤ δ(G)$. If $δ = n - 1$ then $G$ is isomorphic to $K_n - M$ where $M$ is a matching in $K_n$, and $Tγ(G) = 3$ or $4$. If $Tγ(G) = 3$ then $n = 4$. Then $G$ is either $C_4$ or $K_4 - e$. Since $n = 4$ is impossible, we have $n = 4$. Then $G$ is isomorphic to $K_4 - e$ or $C_4$. For these graphs, $κ(G) ≢ n - 3$, which is a contradiction. Hence $δ = n - 3$. Let $X$ be the vertex cut of $G$ with $|X| = n - 3$ and $V - X = \{x_1, x_2, x_3\}$, $X = \{v_1, v_2, ..., v_{n-3}\}$.

Sub Case 3.1. $(V - X) = K_3$

Then every vertex of $V - X$ is adjacent to all the vertices of $X$. Then $\{(V - X) \cup \{v_i\}\} - M$ is a Triple dominating set of $G$ where $M$ is a matching in $K_n$ and
hence $\gamma(G) \leq 4$ gives $n \leq 5$ , since $n \leq 3$ is impossible. We have $n = 4$ or 5. If $n=4$ then $G$ is isomorphic to $K_{1,3}$. Which is a contradiction .If $n=5$ then the graph $G$ has $\gamma(G)=3$ or 4 which is a contradiction to $\kappa(G) = n - 3$.

Subcase 3.2. $(V-X)= K_1 \cup K_2$

Let $x_1, x_2 \in E(G)$ then $x_1$ is adjacent to all the vertices in X and $x_1, x_2$ are not adjacent to at most one vertex in X. If $v_i \in N(x_1) \cup N(x_2)$ then $(\{V-X\} \cup \{v_i\})$ is a Triple dominating set of G and hence $\gamma(G) \leq 4$. This gives $n = 5$. For this graph $\kappa(G)=2$ which is a contradiction. So all $v_i$, either $v_i \in N(x_1)$ or $v_i \in N(x_2)$ or both Then $(V-G) \cup \{v_1\}$ is a Triple dominating set of G by fig:3a. Hence $\gamma(G) \leq 4$ and then $n=4$ or 5. If $n=4$ is impossible , We have $n = 5$. Then $G$ is isomorphic to $C_3$ or $C_4(P_3,0,0)$. But $\kappa[C_3(P_3,0,0)]=1+n-3$ by fig:4c. Hence $G$ is isomorphic to $C_3$.

Case 4. $\gamma(G) = n$ and $\kappa(G) = n - 4$

Then $n-4 \leq \delta$. If $\delta = n-1$ then $G$ is a complete graph which is a contradiction. If $\delta = n-2$ then $G$ is isomorphic to $K_{p,m}$, where $M$ is a matching in $K_p$. Then $\gamma(G)=3$ or 4 then $n=3$ or 4 which is a contradiction to $\kappa(G) = n - 4$. Suppose $\delta = n-3$. Let X be the vertex cut of G with $|X| = n - 4$ and let $X = \{v_1, v_2, ..., v_{n-4}\}$, $V-X = \{x_1, x_2, x_3, x_4\}$. If $(V-X)$ contains an isolated vertex then $\delta \leq n-4$. Which is a contradiction. Hence $(V-X)$ is isomorphic to $K_1 \cup K_2$. Also every vertex of $V-X$ is adjacent to the vertices of X . Let $x_1, x_2, x_3 \in E(G)$. Then $\{x_1, x_2, x_3, x_4\}$ is a triple dominating set of G. Then $\gamma(G) \leq 4$. Hence $n \leq 4$. Which is a contradiction to $\kappa(G) = n - 4$. Thus $\delta(G) = n-4$.

Subcase 4.1. $(V-X) = K_n$

Then every vertex of $V-X$ is adjacent to all the vertices in X. Suppose $E(X) = \emptyset$. Then $|X| \leq 4$ and hence G is isomorphic to $K_{n,4}$ where $S=1,2,3,4$. If $S \neq 1,2$ then $\gamma(G)=3$ or 4 which is a contradiction to $\gamma(G)=n$. Hence G is isomorphic to $K_{1,4}$ or $K_{3,4}$. Suppose $E(X) = \emptyset$. If any one of the vertex in X say $v_i$ is adjacent to all the vertices in X and hence $\gamma(G) \leq 3$ which gives $n \leq 3$ which is a contradiction. Hence every vertex in X is not adjacent to at least one vertices in X then $\{v_1,v_2,v_3,v_4\}$ is a triple dominating set of G and hence $\gamma(G) \leq 4$ then $n \leq 4$. Which is a contradiction to $\kappa(G) = n - 4$.

Subcase 4.2. $(V-X) = P_3 \cup K_1$

Let $x_1$ be the isolated vertex in $(V-X)$ and $(x_1,x_2,x_3)$ be a path then $x_1$ is adjacent to all the vertices in X and $x_2, x_4$ are not adjacent to at most two vertices in X and hence $\{x_1, x_2, x_3, x_4\}$ where $v_i \in E(x_2) \cap X$, $v_2 \in E(x_3) \cap X$ and $v_3 \in E(x_1) \cap X$ is a triple dominating set of G and hence $\gamma(G) \leq 5$ thus $n=5$ then $G$ is isomorphic to $P_3$ or $C_{1,1,1,0}$ or $K_{1,1,1,0}$ or $K_{1,1,1,1}$. All these graph $\gamma(G) \neq n$ by fig:5a. Which is a contradiction.

Subcase 4.3. $(V-X) = K_3 \cup K_1$

Let $x_1, x_2$ be the isolated vertex in $(V-X)$ and $(x_2, x_3, x_4)$ is a complete graph. Then $x_1$ is adjacent to all the vertex in X and $x_2, x_3, x_4$ are not adjacent to at most two vertices in X and hence $\{x_2, x_3, x_4\}$ is a triple dominating set of G and hence $n=5 \, \text{by fig:5b}$. All these graph $\gamma(G) \neq n$.

Subcase 4.4. $(V-X) = K_2 \cup K_2$

Let $x_1, x_2, x_3 \in E(G)$. Since $\delta(G)=n-4$ each $x_i$ is not adjacent to at most one vertex in X then at most one vertex say $v_i \in X$ such that $|N(v_i) \cap (V-X)|=1$. If all $v_i \in X$ such that $|N(v_i) \cap (V-X)| \geq 3$ then $\{x_1, x_2, x_3, x_4\}$ is a triple dominating set of G and hence $n=4$. Which is a contradiction by fig:6a. Then each $x_i$ is non adjacent to at most one vertex in X then at most one vertex say $v_i$ or $v_2 \in X$ such that $|N(v_i) \cap (V-X)|=2$ and $|N(v_2) \cap (V-X)| \geq 3$. It is non adjacent to at most one vertex say $v_i$ or $v_2 \in X$ such that $|N(v_i) \cap (V-X)|=2$ and $|N(v_2) \cap (V-X)| \geq 3$. The converse is obvious.

III. CONCLUSION

In this paper we found an upper bound for the sum of Triple domination number and connectivity of graphs and characterized the corresponding extremal graphs. Similarly Triple domination number with other graph theoretical parameters can be considered.

REFERENCES