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An Empirical Study on Modified ML Estimator and ML Estimator of Weibull Distribution by Using Type II Censored Sample

Farrukh Jamal, Muhammad Arsalan Nasir
Ph. D (Statistics) Scholar in the Islamia University of Bahawalpur, Pakistan
Ph. D (Statistics) Scholar in the Islamia University of Bahawalpur, Pakistan

Abstract
In this paper MML estimator and ML estimator of Weibull distribution by using doubly type II censored sample have been derived and compared in term of asymptotic variances and mean square error. The purpose of conducting the empirical study is to see the closeness of MML estimators to ML estimator, and relative efficiency of censored sample to complete sample.

Keyword--- Type II censored sample, Modified Maximum Likelihood Estimator (MMLE), Maximum Likelihood Estimator (MLE) , Asymptotic Variances, Bias, Mean Square Error, Weibull distribution, Order Statistics.

I. Introduction

When there are censoring present in the sample the Maximum Likelihood Estimator (MLE) can't be found in closed form in general because of intractable term. Dayyeh and Sawi (1996) replace the intractable term of likelihood equations with its expectation, to get the estimators of location parameter by considering the scale parameter equal to 1 of logistic distribution in right Type ll censored sample. Kambo (1978) derived the explicit solution for ML estimator of two parameter exponential distribution in the case of type ll censored sample.

The two-parameter Weibull distribution with scale and shape parameter is given as

$$f(y;\alpha,\lambda)=\frac{\alpha}{\lambda}y^{\alpha-1}\exp(-\frac{y^{\alpha}}{\lambda}) \quad \lambda,\alpha \text{ , y >0} \qquad (1.1)$$

Where α is the shape and λ is the scale parameter .with distribution function

$$F(Y)=\exp\left(-\frac{y^{\alpha}}{\lambda}\right) \qquad (1.2)$$

Related distribution,

Here α is the shape parameter. At α = 1, it reduces to the Exponential distribution, where as at α = 2 , it reduces to the Rayleigh distribution.

II. The Modified Maximum Likelihood Estimator (MMLE) Of The Scale Parameter Of The Weibull Distribution

For the doubly type ll censored sample with r samples censored on the left and s samples censored on the right .The likelihood function is given as

$$L=\frac{n!}{r!s!}\left[F(Y_{r+1})\right]^r\left[1-F(Y_{n-s})\right]^s\prod_{i=r+1}^{n-s}f(y_i) \quad (2.1)$$

Where $r=[nq_1]+1$ and $s=[nq_2]+1$, q_1 is the proportion of left censored sample and q_2 is the proportion of right censored sample .By using (1.1)&(1.2) in (2.1) we get

$$L=\frac{n1}{r!s!}\left[\exp\left(-\frac{y_{r+1}^{\alpha}}{\lambda}\right)\right]^r\left[1-\exp\left(-\frac{y_{n-s}^{\alpha}}{\lambda}\right)\right]^s\prod_{i=r+1}^{n-s}\left[\frac{\alpha}{\lambda}y^{\alpha-1}\exp\left(-\frac{y_i^{\alpha}}{\lambda}\right)\right]$$

The first derivative of the log-likelihood function with respect to λ is given by:

$$\frac{\partial \ln L}{\partial\lambda}=r\left(\frac{y_{r+1}^{\alpha}}{\lambda^2}\right)+\frac{sy_{n-s}}{\lambda^2}\left[\frac{-\exp\left(-\frac{y_{n-s}^{\alpha}}{\lambda}\right)}{1-\exp\left(-\frac{y_{n-s}^{\alpha}}{\lambda}\right)}\right]-\frac{n-s-r}{\lambda}+\frac{1}{\lambda^2}\sum_{i=r+1}^{n-s}y_i^{\alpha}\overset{(set)}{=}0 \qquad (2.2)$$

Put $z_{n-s}=\frac{y_{n-s}^{\alpha}}{\lambda}$ and by using $E\left(\frac{1}{1-\exp(-z_{n-s})}\right)=\frac{n}{n-s-1}$ for intractable term in (2.2) see **Walid Abu-Dayyeh** and **Esam Al Sawi (2006).** Solving (2.2) the modified maximum likelihood estimator of λ is given as

$$\hat{\lambda}=\frac{ry_{r+1}^{\alpha}+sy^{\alpha}{}_{n-s}-\frac{nsy_{n-s}^{\alpha}}{n-s-1}+\sum_{i=r+1}^{n-s}y_i^{\alpha}}{n-s-r} \qquad (2.3)$$

$\lambda^{\wedge}$ from (2.3) is Modified Maximum Likelihood estimator of the scale parameter λ from Weibull distribution of type-ll censored sample.(Also called **Jamal & Nasir estimator**).

III. Asymptotical variance and Bias of scale parameter λ

By using Glivenko-Cantelli lemma given by john and Chen (2003)

We have

$p_1 = G^{-1}(q_1)$ and $p_2 = G^{-1}(1-q_2)$

So as $n \to \infty$

$z_{r+1} = G^{-1}(q_1)$ and $z_{n-s} = G^{-1}(1-q_2)$

z_{r+1} can be evaluated by using following relation

$$G(z_{r+1}) = q_1$$

$$\int_0^{z_{r+1}} f(z)dz = q_1$$

$$z_{r+1} = -\ln(1-q_1) \qquad (3.1)$$

and

$$G(z_{n-s}) = 1-q_2$$

$$\int_{z_{n-s}}^{\infty} f(z)dz = q_2$$

$$z_{n-s} = -\ln(q_2) \qquad (3.2)$$

and

$$\lim E\left\{\frac{1}{n}\sum_{i=r+1}^{n-s} z_i\right\} = \int_{z_{r+1}}^{z_{n-s}} zf(z)dz = \int_{z_{r+1}}^{z_{n-s}} z\exp(-z)dz$$

$n \to \infty$

$$= (1-q_1)(1-\ln(1-q_1)) - q_2(1-\ln q_2) \qquad (3.3)$$

Put $z_{n-s} = \dfrac{y_{n-s}^{\alpha}}{\lambda}$ for intractable term in equation (2.2) and replace intractable term with its expectation

$$\frac{\partial \ln L}{\partial \lambda} = \frac{ry_{r+1}^{\alpha}}{\lambda^2} + \frac{sy_{n-s}^{\alpha}}{\lambda^2} - \frac{nsy_{n-n}^{\alpha}}{\lambda^2(n-s-1)} - \frac{n-s-r}{\lambda} + \frac{1}{\lambda^2}\sum_{i=r+1}^{n-s} y_i^{\alpha}$$

Again differentiate with respect to λ

$$\frac{\partial \ln L}{\partial \lambda^2} = -\frac{2ry_{r+1}^{\alpha}}{\lambda^3} - \frac{2sy_{n-s}^{\alpha}}{\lambda^3} + \frac{2nsy_{n-s}^{\alpha}}{\lambda^3(n-s-1)} + \frac{n-s-r}{\lambda^2} - \frac{2}{\lambda^3}\sum_{i=r+1}^{n-s} y_i^{\alpha}$$

$$= \frac{n}{\lambda^2}\left(-2q_1 z_{r+1} - 2q_2 z_{n-s} + \frac{2q_2 z_{n-s}}{1-q_2-\frac{1}{n}} + 1 - q_1 - q_2 - 2\frac{\sum_{i=r+1}^{n-s} z_i}{n}\right)$$

Where $\therefore q_1 = \dfrac{r}{n} \therefore q_2 = \dfrac{s}{n}$

Applying expectation and multiplying by negative sign on both sides we get For large n i.e $n \to \infty$, substituting the values (3.1) , (3.2) and (3.3) in above equation

$$-E\left(\frac{\partial \ln L}{\partial \lambda^2}\right) = \frac{n}{\lambda^2}\left(\frac{2q_2 \ln q_2}{1-q_2} + 1 - q_1 - q_2 - 2\ln(1-q_1)\right)$$

So Asymptotic variance is given as

$$\operatorname{var}\left(\hat{\lambda}\right) = \frac{1}{-E\left(\frac{\partial \ln L}{\partial \lambda^2}\right)}$$

$$\operatorname{var}\left(\hat{\lambda}\right) = \frac{\lambda^2}{n\left(1-q_1-q_2-2\ln(1-q_1) + \frac{2q_2 \ln q_2}{1-q_2}\right)} \qquad (3.4)$$

From (3.4) we observe as $n \to \infty$, $\operatorname{var}(\lambda^{\wedge}) \to 0$

For Bias put $z_{r+1} = \dfrac{y_{r+1}^{\alpha}}{\lambda}, z_{n-s} = \dfrac{y_{n-s}^{\alpha}}{\lambda}, z_i = \dfrac{y_i^{\alpha}}{\lambda}$ in (2.3) and applying expectation

$$E\left(\hat{\lambda}\right) = \frac{\lambda\left(q_1 z_{r+1} + q_2 z_{n-s} - \frac{q_2}{\left(1-q_1-\frac{1}{n}\right)} z_{n-s} + E\left(\frac{1}{n}\sum_{i=r+1}^{n-s} z_i\right)\right)}{1-q_1-q_2}$$

For large sample size substituting the (3.1) , (3.2) and (3.3) in above equation we get,

$$Bias\left(\hat{\lambda}\right) = E\left(\hat{\lambda}\right) - \lambda = \left(\frac{\frac{q_2 \ln q_2}{1-q_2} - \ln(1-q_1)}{1-q_1-q_2}\right) \qquad (3.5)$$

From (3.5) we observe as $n \to \infty$, $Bias\left(\lambda^{\wedge}\right) \to 0$

IV. Estimation of the Mean Square Error of $\lambda^{\wedge}$ for ordered random variable

The probability density function of $z_{r:n}$ is given by

$$f_{r:n}(z) = \frac{n!}{(r-1)!(n-r)!}(1-\exp(-z))^{r-1}\exp(-(n-r+1))z$$

$$0 \le x \le \infty$$

By definition $E(z_{r:n})$ is given as

$$E(z_{r:n}) = \int_{-\infty}^{\infty} z f_{r:n}(z)dz$$

$$= \int_{0}^{\infty} z \frac{n!}{(r-1)!(n-r)!}(1-\exp(-z))^{r-1}\exp(-(n-r+1))dz$$

Solving the integral and simplifying we get
(See Akhter (2006))

$$E(z_{r:n}) = \sum_{i=1}^{r} \frac{1}{n-j+1} = \alpha_r$$

Similarly $v(z_{r:n}) = \sum_{i=1}^{r} \frac{1}{(n-j+1)^2} = \beta_r$

Now we find mean and variance of $z_r = \frac{y_r^{\alpha}}{\lambda}$

$$E(z_r) = \frac{1}{\lambda} E(y_r^{\alpha})$$

$$E(y_r^{\alpha}) = \lambda \alpha_r \tag{4.1}$$

Where $\alpha_r = \sum_{j=1}^{r} \frac{1}{(n-j+1)}$

$$\operatorname{var}(z_r) = \frac{1}{\left(\overset{2}{\lambda}\right)} \operatorname{var}(y_r^{\alpha})$$

$$\operatorname{var}(y_r^{\alpha}) = \lambda^2 \beta_r \tag{4.2}$$

Where $\beta_r = \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}$

Now by applying expectation on (2.3) and by using (4.1) we get

$$E\left(\hat{\lambda}\right) = \frac{\lambda\left(r\sum_{i=1}^{r+1}\frac{1}{(n-i+1)} + s\sum_{i=1}^{n-s}\frac{1}{(n-i+1)} + \frac{ns}{n-s-1}\sum_{i=1}^{n-s}\frac{1}{(n-i+1)} + \sum_{j=r+1}^{n-s}\sum_{i=1}^{j}\frac{1}{(n-i+1)}\right)}{n-r-s}$$

$$Bais\left(\hat{\lambda}\right) = \frac{\lambda\left(q_1\alpha_{r+1} - \left(\frac{q_2}{1-q_2-\frac{1}{n}} - q_2\right)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)}{1-q_1-q_2} \tag{4.3}$$

Now by applying variance on equation (2.3) and by using (4.2) we get

$$v\left(\hat{\lambda}\right) = \frac{\lambda^2\left(r^2\sum_{i=1}^{r+1}\frac{1}{(n-i+1)^2} + s^2\sum_{i=1}^{n-s}\frac{1}{(n-i+1)^2} - \left(\frac{s}{1-\frac{s}{n}-\frac{1}{n}}\right)^2\sum_{i=1}^{n-s}\frac{1}{(n-i+1)^2} + \sum_{j=1}^{n-s}\sum_{i=1}^{j}\frac{1}{(n-i+1)^2}\right)}{(n-r-s)^2}$$

$$v\left(\hat{\lambda}\right) = \frac{\lambda^2\left(q_1^2\beta_{r+1} + \left(q_2^2 + \left(\frac{q_2}{1-q_2-\frac{1}{n}}\right)^2\right)\beta_{n-s} + \frac{\beta}{n^2}\right)}{(1-q_1-q_2)^2} \tag{4.4}$$

Where $\beta = \sum_{i=r+1}^{n-s} \beta_i \qquad \alpha = \sum_{i=r+1}^{n-s} \alpha_i$

By using (4.3) and (4.4) , we have

$$MSE\left(\lambda^{\wedge}\right) = \frac{\lambda^2}{(1-q_1-q_2)^2}\left(\left(q_1\alpha_{r+1} - \left(\frac{q_2}{1-q_2-\frac{1}{n}} - q_2\right)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)^2 + q_1^2\beta_{r+1} + \left(q_2^2 + \left(\frac{q_2}{1-q_2-\frac{1}{n}}\right)^2\right)\beta_{n-s} + \frac{\beta}{n^2}\right) \tag{4.5}$$

V. The Maximum Likelihood Estimator (MLE)Of The Scale Parameter Of The Weibull Distribution By Using Type II Censored Sample

Kambo (1978) derived the explicit solution for ML estimator of two parameter exponential distribution in the case of doubly type ll censored sample. In this section ML estimator are given for two parameters Weibull distribution in the case of doubly type ll censored sample.

By putting $z_{n-s} = \frac{y_{n-s}^{\alpha}}{\lambda}$ and using (3.2) in (2.2) see **Kambo(1978)** then the maximum likelihood estimator of λ is given as

$$\hat{\lambda} = \frac{ry_{r+1}^{\alpha} - sy^{\alpha}{}_{n-s}\left[\frac{q_2}{1-q_2}\right] + \sum_{i=r+1}^{n-s} y_i^{\alpha}}{n-s-r} \quad (5.1)$$

$\lambda^{\wedge}$ from (5.1) is Maximum Likelihood estimator of the scale parameter λ from Weibull distribution of type-ll censored sample.

For bias put $z_{r+1} = \frac{y_{r+1}^{\alpha}}{\lambda}, z_{n-s} = \frac{y_{n-s}^{\alpha}}{\lambda}, z_i = \frac{y_i^{\alpha}}{\lambda}$ in (5.1) and applying expectation and for large sample size substituting the (3.1) , (3.2) and (3.3) in (5.1) we have

$$Bias\left(\hat{\lambda}\right) = E\left(\hat{\lambda}\right) - \lambda = \left(\frac{\frac{q^2{}_2(\ln q_2)}{1-q_2} + q_2(\ln q_2) - \ln(1-q_1)}{1-q_1-q_2}\right) \quad (5.2)$$

From (5.2) we observe as $n \to \infty$, $Bias\left(\lambda^{\wedge}\right) \to 0$

For asymptotic Variance of $\lambda^{\wedge}$ put $z_{n-s} = \frac{y_{n-s}^{\alpha}}{\lambda}$ and (3.2) for intractable term in equation (2.2) we have

$$\text{var}\left(\hat{\lambda}\right) = \frac{\lambda^2}{n\left(1 - q_1 - q_2 - 2\ln(1-q_1) + \frac{2q_2 \ln q_2}{1-q_2}\right)} \quad (5.3)$$

From (5.3) we observe as $n \to \infty, \text{var}\left(\lambda^{\wedge}\right) \to 0$

For estimation of the mean square error by order random variable by using (4.1) and (4.2) in (5.1) after applying expectation and variance we get,

$$Bias\left(\lambda^{\wedge}\right) = E\left(\lambda^{\wedge}\right) - \lambda = \frac{\lambda\left(q_1\alpha_{r+1} - \left(\frac{q^2{}_2}{1-q_2}\right)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)}{(1-q_1-q_2)} \quad (5.4)$$

and

$$\text{var}\left(\lambda^{\wedge}\right) = \frac{\lambda^2\left(q_1{}^2\beta_{r+1} + \beta_{n-s}\left(\frac{q^2{}_2}{1-q_2}\right)^2 + \frac{\beta}{n^2}\right)}{(1-q_1-q_2)^2} \quad (5.5)$$

From the equation (5.4) and (5.5) we obtain MSE as

$$MSE(\lambda) = \frac{\lambda^2}{(1-q_1-q_2)^2}\left(\left(q_1\alpha_{r+1} - \left(\frac{q^2{}_2}{1-q_2}\right)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)^2 + \left(q_1{}^2\beta_{r+1} + \left(\frac{q^2{}_2}{1-q_2}\right)^2\beta_{n-s} + \frac{\beta}{n^2}\right)\right) \quad (5.6)$$

Corollary: For large sample size estimator from (2.3) and (5.1) become same.

Proof: From (2.3) we have

$$\hat{\lambda} = \frac{ry_{r+1}^{\alpha} + sy^{\alpha}{}_{n-s} - \frac{nsy_{n-s}^{\alpha}}{n-s-1} + \sum_{i=r+1}^{n-s} y_i^{\alpha}}{n-s-r}$$

$$\hat{\lambda} = \frac{ry_{r+1}^{\alpha} + sy^{\alpha}{}_{n-s}\left(1 - \frac{n}{n-s-1}\right) + \sum_{i=r+1}^{n-s} y_i^{\alpha}}{n-s-r}$$

$$\hat{\lambda} = \frac{ry_{r+1}^{\alpha} - sy^{\alpha}{}_{n-s}\left(\frac{q_2 + \frac{1}{n}}{1 - q_2 - \frac{1}{n}}\right) + \sum_{i=r+1}^{n-s} y_i^{\alpha}}{n-s-r}$$

For large n estimator from (2.3) and (5.1) are exactly same that why asymptotic variances, MSEs (for large sample size) are same, this establishes the Corollary.

VI. Comparison of Censored Sample To Complete Sample.

In this section we compare the reduction in efficiency of censored sample to the complete sample for the scale parameter of Weibull distribution keeping the shape parameter fixed. Now we find estimator of complete sample using method of maximum likelihood from the equation (1.1) we get.

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} y_i^{\alpha}}{n} \quad (6.1)$$

$\hat{\lambda}$ from (6.1) is the estimator of compete sample.

For estimation of the mean square error by order random variable by using (4.1) and (4.2) in (6.1) after applying expectation and variance we get,

$$Bias\left(\hat{\lambda}\right) = \lambda\left(\frac{\alpha}{n} - 1\right) \quad (6.2)$$

and

$$\text{var}\left(\hat{\lambda}\right) = \frac{\lambda^2 \beta}{n^2} \quad (6.3)$$

From (6.2) and (6.3) we obtained MSE as,

$$MSE\left(\hat{\lambda}\right)=\lambda^{2}\left(\left(\frac{\alpha}{n}-1\right)^{2}+\left(\frac{\beta}{n^{2}}\right)\right) \qquad (6.4)$$

It is the MSE of complete sample

Reduction in efficiency =(MSE of censored sample /MSE of complete sample)-1

Which can be obtain from (4.5) and (6.4)

(6.5)

VII. Discussion and Conclusion.

1) In this paper a simple approximation has been proposed for intractable term (See Walid Abu-Dayyeh and Esam Al Sawi 2006) , to estimate scale parameter keeping shape parameter fixed of two parameter Weibull distribution from doubly type ll censored sample.

2) The table(1) of asymptotic variance for n = 10 ,20 ,30 ,50 &100 and table(2) of MSE by using theory of order statistics for n = 10 , 20 , 30, 50 &100 are given for the MML estimator and ML estimator with maximum 0.6 total proportion of censored sample i.e $q_1+q_2=0.6$, where q_1 is the proportion of left censored sample and q_2 is the proportion of right censored sample. For any sample size there are three conditions for q_1 and q_2. (i) $q_1 > q_2$ (ii) $q_1 < q_2$ (iii) $q_1 = q_2$.

3) From **Table (1),** as the sample size increases the asymptotic variances are decreased with the same proportion of left and right censored sample. Asymptotic variances depend on total proportion of censored sample, as total proportion increases. the variances also increases. Asymptotic variances of MML estimator exactly same as variances of ML estimators.

4) From **Table (2)** for $q_1=q_2=0$. no censoring scheme is involved in other words there is no missing element in the sample then the MSE,s of MML estimator and ML estimator tends to decreases for any sample of size n. As the sample size increases the Mean square errors of MML estimator tends to decreases but Mean square errors of ML estimators increases.

5) The purpose of conducting **empirical study** is to see the closeness of MML estimators to ML estimator of Weibull distribution for type II censored sample .Since it has been seen that the asymptotic variances of Modify Maximum Likelihood (MML) estimator are exactly same to Maximum Likelihood (ML) estimator. MSE,s of MML estimator is minimum as compare to ML estimator for small sample size . So for small sample size MML estimator is better than ML estimator. For large sample size of n (n>100) MSE,s of both MML estimator and ML estimator are same.

6) From **Table (3)** we observe that for $q_1=q_2=0$ as the sample size increases the reduction in efficiency decreases. And for rest of values the sample size increases the reduction in efficiency also increases with the same proportion of left and right censored sample. As the total proportion of censored sample increases , the reduction in efficiency also increases.

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Table 1 : The asymptotic variance of MML& ML estimates from doubly censored sample in term of λ^2 for n=10, 20, 30, 50 and100 from the equation (3.4) and (5.3)

q1	q2	n=10		n=20		n=30	
		$\mathrm{var}(\lambda_1)/\lambda^2$	$\mathrm{var}(\lambda_2)/\lambda^2$	$\mathrm{var}(\lambda_1)/\lambda^2$	$\mathrm{var}(\lambda_2)/\lambda^2$	$\mathrm{var}(\lambda_1)/\lambda^2$	$\mathrm{var}(\lambda_2)/\lambda^2$
0	**0.1**	**0.2575**	0.2575	**0.1288**	0.1288	**0.0858**	0.0858
0.1	**0.1**	**0.2004**	0.2004	**0.1002**	0.1002	**0.0668**	0.0668
0.1	**0.2**	**0.9434**	0.9434	**0.4717**	0.4717	**0.3145**	0.3145
0.2	**0.1**	**0.1576**	0.1576	**0.0788**	0.0788	**0.0525**	0.0525
0.2	**0.2**	**0.414**	0.414	**0.207**	0.207	**0.138**	0.138
0.3	**0.1**	**0.1247**	0.1247	**0.0624**	0.0624	**0.0416**	0.0416
0.3	**0.2**	**0.2447**	0.2447	**0.1224**	0.1224	**0.0816**	0.0816
0.3	**0.3**	**1.2289**	1.2289	**0.6145**	0.6145	**0.4096**	0.4096
0.4	**0.1**	**0.099**	0.099	**0.0495**	0.0495	**0.033**	0.033
0.4	**0.2**	**0.1621**	0.1621	**0.081**	0.081	**0.054**	0.054

q1	q2	n=50		n=100	
		$\mathrm{var}(\lambda_1)/\lambda^2$	$\mathrm{var}(\lambda_2)/\lambda^2$	$\mathrm{var}(\lambda_1)/\lambda^2$	$\mathrm{var}(\lambda_2)/\lambda^2$
0	**0.1**	**0.0515**	0.0515	**0.0258**	0.0258
0.1	**0.1**	**0.0401**	0.0401	**0.02**	0.02
0.1	**0.2**	**0.1887**	0.1887	**0.0943**	0.0943
0.2	**0.1**	**0.0315**	0.0315	**0.0158**	0.0158
0.2	**0.2**	**0.0828**	0.0828	**0.0414**	0.0414
0.3	**0.1**	**0.0249**	0.0249	**0.0125**	0.0125
0.3	**0.2**	**0.0489**	0.0489	**0.0245**	0.0245
0.3	**0.3**	**0.2458**	0.2458	**0.1229**	0.1229
0.4	**0.1**	**0.0198**	0.0198	**0.0099**	0.0099
0.4	**0.2**	**0.0324**	0.0324	**0.0162**	0.0162

Where $\mathrm{var}(\lambda_1)/\lambda^2$ **is the variance of MML estimator from (3.4)**
and $\mathrm{var}(\lambda_2)/\lambda^2$ **is the variance of ML estimator from (5.3)**

Table 2 :The MSE of MML estimates from doubly censored sample in term of λ^2 for n=10, 20, 30, 50 and 100 from the equation (4.5) and (5.6)

q1	q2	n=10		n=20		n=30	
		$MSE(\lambda_1)/\lambda^2$	$MSE(\lambda_2)/\lambda^2$	$MSE(\lambda_1)/\lambda^2$	$MSE(\lambda_2)/\lambda^2$	$MSE(\lambda_1)/\lambda^2$	$MSE(\lambda_2)/\lambda^2$
0	**0**	**0.0293**	0.0448	**0.009**	0.013	**0.0044**	0.0062
0	**0.1**	**0.0792**	0.1024	**0.0749**	0.0849	**0.0757**	0.0822
0	**0.2**	**0.2244**	0.2968	**0.2341**	0.2695	**0.2394**	0.263
0	**0.3**	**0.4778**	0.6618	**0.5059**	0.5951	**0.5173**	0.5765
0	**0.4**	**0.9125**	1.3403	**0.9691**	1.1708	**0.9904**	1.1228
0	**0.5**	**1.6975**	2.7035	**1.8027**	2.2576	**1.8408**	2.1358
0	**0.6**	**3.2551**	5.8783	**3.4522**	4.5644	**3.5224**	4.2307
0.1	**0**	**0.0516**	0.0707	**0.0256**	0.0306	**0.0197**	0.0219
0.1	**0.1**	**0.0453**	0.0653	**0.0343**	0.0424	**0.0333**	0.0385
0.1	**0.2**	**0.1565**	0.228	**0.1626**	0.1977	**0.1671**	0.1907
0.1	**0.3**	**0.3951**	0.5972	**0.4256**	0.5243	**0.4385**	0.5041
0.1	**0.4**	**0.8628**	1.3786	**0.9349**	1.1792	**0.9623**	1.123
0.1	**0.5**	**1.8336**	3.1812	**1.9858**	2.5961	**2.0413**	2.4374
0.2	**0**	**0.1347**	0.159	**0.0972**	0.1035	**0.0883**	0.0911
0.2	**0.1**	**0.0406**	0.0531	**0.0141**	0.0178	**0.0084**	0.0106
0.2	**0.2**	**0.0884**	0.1503	**0.0813**	0.1121	**0.0822**	0.1031
0.2	**0.3**	**0.2772**	0.4901	**0.3013**	0.4067	**0.3132**	0.3838
0.2	**0.4**	**0.7451**	1.375	**0.8303**	1.1312	**0.864**	1.0627

0.3	**0**	**0.3622**	0.3938	**0.2985**	0.3066	**0.2821**	0.2858
0.3	**0.1**	**0.1386**	0.1343	**0.0737**	0.0682	**0.0563**	0.0523
0.3	**0.2**	**0.084**	0.1148	**0.0349**	0.0513	**0.0235**	0.0351
0.3	**0.3**	**0.1648**	0.3576	**0.1467**	0.2462	**0.1474**	0.2149

q1	**q2**	**n=50**		**n=100**	
		$MSE(\lambda_1)/\lambda^2$	$MSE(\lambda_2)/\lambda^2$	$MSE(\lambda_1)/\lambda^2$	$MSE(\lambda_2)/\lambda^2$
0	**0**	**0.0018**	0.0024	**0.0005**	0.0006
0	**0.1**	**0.0771**	0.081	**0.0788**	0.0807
0	**0.2**	**0.2443**	0.2586	**0.2485**	0.2556
0	**0.3**	**0.5272**	0.5627	**0.5351**	0.5528
0	**0.4**	**1.0083**	1.0869	**1.0222**	1.0612
0	**0.5**	**1.8725**	2.0459	**1.8969**	1.9824
0	**0.6**	**3.5802**	3.991	**3.6244**	3.825
0.1	**0**	**0.0162**	0.0171	**0.0145**	0.0147
0.1	**0.1**	**0.0335**	0.0365	**0.0342**	0.0357
0.1	**0.2**	**0.1717**	0.1859	**0.1756**	0.1828
0.1	**0.3**	**0.4498**	0.4892	**0.4588**	0.4786
0.1	**0.4**	**0.9854**	1.081	**1.0035**	1.051
0.1	**0.5**	**2.0874**	2.3205	**2.1229**	2.2379
0.2	**0**	**0.0827**	0.0837	**0.0797**	0.0799
0.2	**0.1**	**0.0051**	0.0064	**0.0033**	0.004
0.2	**0.2**	**0.0842**	0.0969	**0.0864**	0.0929
0.2	**0.3**	**0.3242**	0.3668	**0.3333**	0.3547
0.2	**0.4**	**0.8929**	1.0115	**0.9157**	0.9748
0.3	**0**	**0.2713**	0.2726	**0.2646**	0.265
0.3	**0.1**	**0.044**	0.0416	**0.0357**	0.0345
0.3	**0.2**	**0.0162**	0.0236	**0.0118**	0.0157
0.3	**0.3**	**0.1503**	0.1916	**0.1538**	0.1748

Where $MSE(\lambda_1)/\lambda^2$ **is the variance of MML estimator from (4.5)**
And $MSE(\lambda_2)/\lambda^2$ **is the variance of ML estimator from (5.6)**

Tables 3: Comparison of censored sample as compare to complete sample for n=10,20,30 ,50 and 100 from the equation (6.5)

q1	***q2***	$R(\lambda)-1$	$R(\lambda)-1$	$R(\lambda)-1$	$R(\lambda)-1$	$R(\lambda)-1$
		n=10	***n=20***	***n=30***	***n=50***	***n=100***
0	**0**	0.5291	0.4437	0.4035	0.3612	0.3152
0	**0.1**	2.4954	8.4363	17.5136	44.0342	154.6302
0	**0.2**	9.1344	28.9636	58.2484	142.6696	491.7705
0	**0.3**	21.5939	65.1638	128.8755	311.6461	1.06E+03
0	**0.4**	44.7613	129.17	251.9537	602.9446	2.04E+03
0	**0.5**	91.3013	249.9993	480.1609	1.14E+03	3.82E+03
0	**0.6**	199.6956	506.4781	952.0908	2.22E+03	7.37E+03
0.2	**0.4**	45.9452	124.7635	238.4038	561.0169	1.88E+03

0.3	**0**	12.4445	33.088	63.3821	150.4666	509.7857
0.3	**0.1**	3.5845	6.5822	10.7856	22.1036	65.5889
0.3	**0.2**	2.9185	4.7021	6.9097	12.1289	29.3335
0.3	**0.3**	11.2106	26.3692	47.4243	105.4582	336.058

0.1	**0**	1.4134	2.3977	3.9426	8.4748	27.3511
0.1	**0.1**	1.2306	3.7106	7.664	19.3043	67.8823
0.1	**0.2**	6.7839	20.9797	41.9504	102.3036	351.4054
0.1	**0.3**	19.39	57.2923	112.5756	270.8252	921.5724
0.1	**0.4**	46.0683	130.1017	251.9959	599.667	2.03E+03
0.1	**0.5**	107.6122	287.639	548.1027	1.29E+03	4.31E+03
0.2	**0**	4.427	10.5036	19.5165	45.5219	153.0383
0.2	**0.1**	0.8121	0.9736	1.3917	2.5573	6.7338
0.2	**0.2**	4.1325	11.4598	22.2167	52.8639	178.065
0.2	**0.3**	15.732	44.2185	85.4566	202.7927	682.7043