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Connectedness in Ideal Bitopological Spaces

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Abstract. In this paper we study the notion of connectedness in ideal bitopological spaces.
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1. Introduction

In 1961 Kelly [7] introduced the concept of bitopological spaces as an extension of topological spaces. A bitopological space (X, $□_1$, $□_2$) is a nonempty set X equipped with two topologies $□_1$ and $□_2$ [7]

The notion of ideal in topological spaces was studied by Kuratowski [8] and Vaidyanathaswamy [13]. Applications to various fields were further investigated by Jankovic and Hamlett [6]; Dontchev [2]; Mukherjee [9]; Arenas [1]; Navaneethakrishnan [11]; Nasef and Mahmoud [10], etc.

The purpose of this paper is to introduce and study the notion of connectedness in ideal bitopological spaces. We study the notions of pairwise *-connected ideal bitopological spaces, pairwise *-separated sets, pairwise **s*-connected sets and pairwise *-connected sets in ideal bitopological spaces.

2. Preliminaries

An ideal $\mathcal{I}$ on a topological space (X, □) is a nonempty collection of subsets of *X* which satisfies

i. A $\in \mathcal{I}$ and B $\subset$ A $\Rightarrow$ B $\in \mathcal{I}$ and
ii. A $\in \mathcal{I}$ and B $\in \mathcal{I} \Rightarrow$ A $\cup$ B $\in \mathcal{I}$

An ideal topological space is a topological space (X, □) with an ideal $\mathcal{I}$ on X, and is denoted by (X, □, $\mathcal{I}$). Given an ideal topological space (X, □, $\mathcal{I}$) and If $\mathcal{P}$(X) is the set of all subsets of X, a set operator,
$(.)^*: \mathcal{P}(X) \to \mathcal{P}(X)$ is called the local mapping [7] of A with respect to □ and $\mathcal{I}$ and is defined as follows: For A $\subset$ X $A^*(□, \mathcal{I})$ = {x∈X |U ∩ A ∉ $\mathcal{I}$, ∀ U∈ □, where x∈U}.

A Kuratowski closure operator Cl* (.) for a topology τ* (□ ,$\mathcal{I}$), called the *-topology, finer than τ, is defined by Cl*(A) = A ∪ A* (□ , $\mathcal{I}$) [6]. Without ambiguity, we write A* for A* (□ , $\mathcal{I}$) and τ* for τ* (□ , $\mathcal{I}$). For any ideal space (X, □, $\mathcal{I}$), the collection {V\J: V ∈ □ and J ∈ $\mathcal{I}$} is a basis for □*.

Definition 2.1. [3] An ideal topological space (X, □, $\mathcal{I}$)
is called *-connected [3] if X cannot be written as the disjoint union of a nonempty open set and a nonempty *-open set.

Recall that [6] if (X, τ, I) is an ideal topological space and A is a subset of X, then (A, τ_A $\mathcal{I}_A$), where τ_A is the relative topology on A and $\mathcal{I}_A$ = {A ∩ J: J ∈ $\mathcal{I}$} is an ideal topological space

Definition 2.2. [3] A subset A of an ideal topological space (X, □, $\mathcal{I}$) is called *-connected if (A, τ_A, $\mathcal{I}_A$) is *-connected.

Lemma 2.1. [6] Let (X, □, $\mathcal{I}$) be an ideal topological space and B ⊂ A ⊂ X. Then, B* (τ_A, $\mathcal{I}_A$) =
B* (τ, $\mathcal{I}$) ∩ A.

Lemma 2.2. [4] Let (X, □, $\mathcal{I}$) be an ideal topological space and B ⊂ A ⊂ X. Then Cl^*_A(B) = Cl*(B) ∩ A.

Definition 2.3. [3] A subset A of an ideal space (X, □, $\mathcal{I}$) is said to be *-dense [2] if Cl*(A) = X. An ideal space (X, □, $\mathcal{I}$ is said to be [3] *-hyperconnected if A is *-dense for every open subset A ≠ ϕ of X.

Lemma 2.3. [2] Let (X, □, $\mathcal{I}$) be an ideal topological space. For each V ∈ □ *, τ^*_V = $(\tau_V)^*$.

Lemma 2.4. [3] Let (X, □, $\mathcal{I}$) be a topological space, A ⊂ Y ⊂ X and Y ∈ τ. Then A is *-open in Y is equivalent to A is *-open in X
Proof: A is *-open in Y ⇒ A is *-open in X. Since Y ∈ τ ⊂ τ* by Lemma 6, A is *-open in X. A is *-open in X ⇒ A is *-open in Y for if A is *-open in X. By Lemma 6, A = A ∩ Y is *-open in Y.

Definition 2.4. [7] A bitopological space (X, $□_1$, $□_2$, $\mathcal{I}$ is an ideal bitopological space where $\mathcal{I}$ is defined on a bitopological space (X, $□_1$, $□_2$).

Throughout the present paper, (X, τ_1, τ_2, $\mathcal{I}$ will denote a bitopological space with no assumed separation properties. For a subset A of a bitopological space (X, τ_1, τ_2, $\mathcal{I}$), Cl(A) and Int(A) will denote the closure and interior of A in (X,τ_1,τ_2, $\mathcal{I}$, respectively.

3. Connectedness in Ideal Bitopological Spaces

Definition 3.1. An ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is called pairwise *-connected [3] if X cannot be written as the disjoint union of a nonempty τ_i open set and a nonempty τ_j^* -open set. {i , j = 1, 2; i ≠ j}

Remark 3.1. Since every τ_i open (τ_j open) set is τ_i^* (respectively τ_j^* open). It follows that every pairwise *-connected ideal bitopological space is pairwise connected but the converse may not be true.

Definition 3.2. [3] An ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is said to be pairwise hyperconnected if A is τ_j^* dense for every τ_i open set A ≠ ∅ of X

Definition 3.3. A subset A of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is called pairwise *-connected if (A, $(\tau_1)_A$, $(\tau_2)_A$, I_A) is pairwise *-connected.

Definition 3.4. Nonempty subsets A, B of an ideal bitopological space (X, τ_1, τ_2 $\mathcal{I}$) are called pairwise *-separated if τ_iCl*(A) ∩ B = A ∩ τ_jCl(B) = ϕ.

Theorem 3.1. Let (X, τ_1, τ_2, $\mathcal{I}$) be an ideal bitopological space. If A, B are pairwise *-separated sets of X and A ∪ B ∈ τ_1 ∩ τ_2 then A is τ_i open and B is τ_j^*-open. {i , j = 1 , 2; i ≠ j}

Proof: Since A and B are pairwise *-separated in X, then B = (A ∪ B) ∩ (X - Cl*(A)). Since A∪ B is biopen and τ_jCl*(A) is τ_j^* -closed in X, B is τ_j^*-open in X. Similarly A = (A ∪ B) ∩ (X - Cl*(B)) and we obtain that A is $\tau_i$open in X.

Theorem 3.2. Let (X, τ_1, τ_2 $\mathcal{I}$) be an ideal bitopological space and A, B ⊂ Y ⊂ X. Then A and B are pairwise *-separated in Y if and only if A, B are pairwise *-separated in X

Proof: It follows from Lemma 2 that τ_iCl*(A) ∩ B = A ∩ τ_jCl(B) = ϕ.

Theorem 3.3. If f: (X, τ_1, τ_2, $\mathcal{I}$) → (Y, τ_1, τ_2) is a pairwise continuous onto mapping. Then if (X, τ_1, τ_2, $\mathcal{I}$) is a pairwise *-connected ideal bitopological space (Y, τ_1, τ_2) is also pairwise connected.

Proof: It is known that connectedness is preserved by continuous surjections. Hence every pairwise *-connected space is connected and the proof is obvious.

Definition 3.5. A subset A of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is called pairwise *s-connected if A is not the union of two pairwise *-separated sets in (X, τ_1, τ_2, $\mathcal{I}$)

Theorem 3.4. Let Y be a biopen subset of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) {i , j = 1 , 2; i ≠ j} The following are equivalent:

i. Y is pairwise *s-connected in (X, τ_1, τ_2, $\mathcal{I}$)
ii. Y is pairwise *-connected in (X, τ_1, τ_2, $\mathcal{I}$)

Proof: : i) ⇒ ii) Let Y be pairwise *s-connected in (X, τ_1, τ_2, $\mathcal{I}$) and suppose that Y is not pairwise *-connected in (X, τ_1, τ_2, $\mathcal{I}$). There exist non empty disjoint τ_i open set A, in Y and τ_j^* open set B in Y s.t Y = A ∪ B. Since Y is open in X, by Lemma 2.4 A and B are $\tau_i$open and τ_j^* open in X, respectively. Since A and B are disjoint, then τ_iCl*(A) ∩ B = ∅ = A ∩ τ_jCl(B). This implies that A, B are pairwise *-separated sets in X. Thus, Y is not pairwise *s-connected in (X, τ_1, τ_2, $\mathcal{I}$). Hence we arrive at a contradiction and Y is pairwise *-connected in (X, τ_1 , τ_2, $\mathcal{I}$).

ii) ⇒ i) Suppose Y is not pairwise *s-connected in (X, τ_1, τ_2, $\mathcal{I}$). There exist two pairwise *-separated sets A, B s.t Y = A ∪ B. By Theorem 3.1, A and B are τ_i open and τ_j -open in Y, respectively {i , j = 1 , 2; i ≠ j}. By Lemma 2.4, A and B are τ_i open and τ_j^* -open in X respectively. Since A and B are *-separated in X, then A and B are nonempty and disjoint. Thus, Y is not pairwise *-connected. This is a contradiction.

Theorem 3.5. Let (X, τ_1, τ_2, $\mathcal{I}$) be an ideal bitopological space. If A is a pairwise *s-connected set of X and H, G are pairwise *-separated sets of X with A ⊂ H ∪ G, then either A ⊂ H or A ⊂ G. {i , j = 1 , 2; i ≠ j}

Proof: Let A ⊂ H ∪ G. Since A = (A ∩ H) ∪ (A ∩ G), then (A ∩ G) ∩ τ_iCl*(A ∩ H) ⊂ G ∩ Cl*(H) = ∅. By similar reasoning, we have (A ∩ H) ∩ τ_jCl(A ∩ G) ⊂ H ∩ Cl*(G) = ∅. Suppose that A ∩ H and A ∩ G are nonempty. Then A is not pairwise *s-connected. This is a contradiction. Thus, either A ∩ H = ∅ or A ∩ G = ∅ This implies that A ⊂ H or A ⊂ G

Theorem 3.6. If A is a pairwise *s-connected set of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) and

$A \subset B \subset \tau_i Cl^*(A) \cap \tau_j Cl(B)$ then B is pairwise *s-connected $\{i, j = 1, 2;\ i \neq j\}$.

Proof: Suppose B is not pairwise *s-connected. There exist pairwise *-separated sets H and G of X such that $B = H \cup G$. This implies that H and G are nonempty and $\tau_i Cl^*(H) \cap G = H \cap \tau_j Cl(G) = \phi$. By Theorem 3.5, we have either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $Cl^*(A) \subset Cl^*(H)$ and $G \cap Cl^*(A) = \phi$ This implies that $G \subset B \subset Cl^*(A)$ and $G = Cl^*(A) \cap G = \phi$. Thus, G is an empty set for if G is nonempty, this is a contradiction. Suppose that $A \subset G$. By similar way, it follows that H is empty. This is a contradiction. Hence, B is pairwise *s-connected.

Corollary 3.1. If A is a pairwise*s-connected set in an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$) then $\tau_i Cl^*(A)$ is pairwise *s-connected.

Theorem 3.7. If $\{M_i : i \in N\}$ is a nonempty family of pairwise *s-connected sets of an ideal space $(X, \tau_1, \tau_2, \mathcal{I})$ with $\bigcap_{i \in I} Mi \neq \phi$ Then $\bigcup_{i \in I} Mi$ is pairwise *s-connected.

Proof: Suppose that $\bigcup_{i \in I} Mi$ is not pairwise *s-connected. Then we have $\bigcup_{i \in I} Mi = H \cup G$, where H and G are pairwise *-separated sets in X. Since $\bigcap_{i \in I} Mi \neq \phi$ we have a point x in $\bigcap_{i \in I} Mi$
Since $x \in \bigcup_{i \in I} Mi$, either $x \,\varepsilon\, H$ or $x \,\varepsilon\, G$. Suppose that $x \,\varepsilon\, H$. Since $x \,\varepsilon\, M_i$ for each $i \,\varepsilon\, N$, then M_i and H intersect for each $i \,\varepsilon\, N$. By theorem 3.5; $M_i \subset H$ or $M_i \subset G$. Since H and G are disjoint, $M_i \subset H$ for all $i \,\varepsilon\, Z$ and hence $\bigcup_{i \in I} Mi \subset H$. This implies that G is empty. This is a contradiction. Suppose that $x \,\varepsilon\, G$. By similar way, we have that H is empty. This is a contradiction. Thus, $\bigcup_{i \in I} Mi$ is pairwise *s-connected.

Theorem 3.8. Suppose that $\{M_n: n \,\varepsilon\, N\}$ is an infinite sequence of pairwise *-connected open sets of an ideal space $(X, \tau_1, \tau_2, \mathcal{I})$ and $M_n \cap M_{n+1} \neq \phi$ for each $n \,\varepsilon\, N$. Then $\bigcup_{i \in I} Mi$ is pairwise*connected.

Proof: By induction and Theorems 3.4 and 3.7, the set $N_n = \bigcup_{k \leq n} Mk$ is a pairwise *-connected open set for each $n \,\varepsilon\, N$. Also, N_n have a nonempty intersection. Thus, by Theorems 13 and 17, $\bigcup_{n \in N} Mn$ is pairwise *-connected

Definition 3.6. Let X be an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ and $x \in X$. The union of all pairwise *s-connected subsets of X containing x is called the pairwise *-component of X containing x.

Theorem 3.9. Each pairwise *-component of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is a maximal pairwise *s connected set of X.

Theorem 3.10. The set of all distinct pairwise *-components of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ forms a partition of X

Proof: Let A and B be two distinct pairwise *-components of X. Suppose that A and B intersect. Then, by Theorem 3.7, $A \cup B$ is pairwise *s-connected in X. Since $A \subset A \cup B$, then A is not maximal. Thus, A and B are disjoint.

Theorem 3.11. Each pairwise*-component of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise *-closed in X.

Proof: Let A be a pairwise *-component of X. By Corollary 3.1, $\tau_i Cl^*(A)$ is pairwise*s-connected and $A = \tau_i Cl^*(A)$. Thus, A is pairwise *-closed in X.

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