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Implicit Finite Difference Solution of Boundary Layer Heat Flow over a Flat Plate

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ABSTRACT
In this paper, boundary layer heat flow over a flat plate is discussed. Similarity transformation is employed to transform the governing partial differential equations into ordinary ones, which are then solved numerically using implicit finite difference scheme namely Keller box method. The obtained Keller box solutions, in comparison with the previously published work are performed and are found to be in a good agreement. Numerical results for the temperature distribution have been shown graphically for different values of the Prandtl number.
Keywords - Boundary layer flow, Convection of heat, Keller box method, nonlinear differential equation.

I. INTRODUCTION

The mathematical complexity of convection heat transfer is found to the non-linearity of the Navier-Stokes equations of motion and the coupling of flow and thermal fields. The boundary layer concept, first introduced by Prandtl in 1904, provides major simplifications. This concept is based on the belief that under special conditions certain terms in the governing equations are much smaller than others and therefore can be neglected without significantly affecting the accuracy of the solution.

Most boundary-layer models can be reduced to systems of nonlinear ordinary differential equations which are usually solved by numerical methods. In this paper, solution of boundary layer heat flow over a flat plate is study with the help of implicit finite difference Keller box method. The same problem is solved by M. Esameilpour et al. [1] with the help of He's Homotopy perturbation method. H. Mirgolbabaei [2] solved using Adomian Decomposition Method (ADM).

II. GOVERNING EQUATIONS

The boundary layer equations assume the following: (i) steady, incompressible flow, (ii) laminar flow, (iii) no significant gradients of pressure in the x-direction, and (iv) velocity gradients in the x-direction are small compared to velocity gradients in the y-direction. The reduced Navier–Stokes equations for boundary layer over flat plate [3]

Continuity Equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

Momentum Equation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta\left(T-T_\infty\right) \tag{2}$$

Under a boundary layer assumption, the energy transport equation is also simplified.

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 u}{\partial y^2} \tag{3}$$

Subject to the boundary conditions

$$\begin{aligned}
&\text{i) No slip } u=0\\
&\text{ii) Impermeability } v=0\\
&\text{iii) Wall Temperature } T=T_w\\
&\text{iv) Uniform flow } u=U_\infty\\
&\text{v) Uniform flow } v=0\\
&\text{vi) Uniform Temperature } T=T_\infty
\end{aligned} \tag{4}$$

The solution to the momentum equation is independent of the energy solution. However, the solution of the energy equation is still depends on the momentum solution.

The following dimensionless variables are introduced in the transformation [1]

$$\eta=\frac{y}{\sqrt{x}}\mathrm{Re}_x^{0.5}$$

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}$$

where θ is non -dimensional form of the temperature and the Reynolds number is defined as

$$\mathrm{Re}=\frac{u_\infty x}{\nu}$$

the governing equations (1)-(3) can be reduced to two equations, where f is a function of the similarity variable η

$$f''' + \frac{1}{2} ff' = 0 \tag{5}$$

$$\varepsilon\theta'' + \frac{1}{2} f\theta' = 0 \tag{6}$$

where $\varepsilon = \frac{1}{\Pr}$ and f is related to the u velocity by $f' = \frac{u}{u_\infty}$.

The corresponding boundary conditions are

$$f(0)=0, \quad f'(0)=0, \quad \theta(0)=1, \qquad f'(\infty)=1, \quad \theta(\infty)=0. \tag{7}$$

III. KELLER BOX METHOD

Equation (5) and (6) subject to the boundary conditions (7) is solved numerically using implicit finite difference method that is known as Keller-box in combination with the Newton's linearization techniques as described by Cebeci and Bradshaw [4]. This method is completely stable and has second-order accuracy.

In this method the transformed differential equations (5)-(6) are writes in terms of first order system, for that introduce new dependent variable u, v, w such that

$$f' = u \qquad u' = v \qquad \theta' = w \tag{8}$$

where prime denotes the differentiation w.r.to η.

Equation (5) and (6) become

$$v' + \frac{1}{2} fv = 0 \tag{9}$$

$$\varepsilon w' + \frac{1}{2} fw = 0 \tag{10}$$

with new independent boundary conditions are

$$f(0) = 0, u(0) = 0, \theta(0) = 1 \qquad u(\infty) = 1, \theta(\infty) = 0 \tag{11}$$

Now write the finite difference approximations of the ordinary differential equations (8) for the midpoint $\left(x^n, \eta_{j-\frac{1}{2}}\right)$ of the segment using centered difference derivatives, this is called centering about $\left(x^n, \eta_{j-\frac{1}{2}}\right)$.

$$\frac{f_j^n - f_{j-1}^n}{h_j} = \frac{u_j^n + u_{j-1}^n}{2} \tag{12}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = \frac{v_j^n + v_{j-1}^n}{2} \tag{13}$$

$$\frac{\theta_j^n - \theta_{j-1}^n}{h_j} = \frac{w_j^n + w_{j-1}^n}{2} \tag{14}$$

Ordinary differential equations (9) and (10) are approximated by the centering about the mid-point $\left(x^{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}}\right)$ of the rectangle.

$$\left(\frac{v_j - v_{j-1}}{h_j}\right) + \frac{1}{2}\left[\frac{f_j + f_{j-1}}{2}\right]\left[\frac{v_j + v_{j-1}}{2}\right] = -\left[\frac{v_j - v_{j-1}}{h_j} + \frac{1}{2}(fv)\right]_{j-\frac{1}{2}}^{n-1} \tag{15}$$

$$\varepsilon\left(\frac{w_j - w_{j-1}}{h_j}\right) + \frac{1}{2}\left[\frac{f_j + f_{j-1}}{2}\right]\left[\frac{w_j + w_{j-1}}{2}\right] = -\left[\varepsilon\frac{w_j - w_{j-1}}{h_j} + \frac{1}{2}(fw)\right]_{j-\frac{1}{2}}^{n-1} \tag{16}$$

Now linearize the nonlinear system of equations (12)-(16) using the Newton's quasi-linearization method [5]

$$\begin{aligned}
\delta f_j - \delta f_{j-1} - \frac{h_j}{2}\left(\delta u_j + \delta u_{j-1}\right) &= (r_1)_j \\
\delta u_j - \delta u_{j-1} - \frac{h_j}{2}\left(\delta v_j + \delta v_{j-1}\right) &= (r_2)_j \\
\delta \theta_j - \delta \theta_{j-1} - \frac{h_j}{2}\left(\delta w_j + \delta w_{j-1}\right) &= (r_3)_j
\end{aligned} \tag{17}$$

$$(a_1)_j \delta v_j + (a_2)_j \delta v_{j-1} + (a_3)_j \delta f_j + (a_4)_j \delta f_{j-1} = (r_4)_j$$

$$(b_1)_j \delta w_j + (b_2)_j \delta w_{j-1} + (b_3)_j \delta f_j + (b_4)_j \delta f_{j-1} = (r_5)_j$$

The linearized difference equation of the system (17) has a block tridiagonal structure. In a vector matrix form, it can be written as

$$\begin{bmatrix} [A_1] & [C_1] & & & & & \\ [B_2] & [A_2] & [C_2] & & & & \\ & [B_3] & [A_3] & [C_3] & & & \\ & & & : & & : & \\ & & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\ & & & & & [B_J] & [A_J] \end{bmatrix} \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ [\delta_3] \\ : \\ : \\ [\delta_J] \end{bmatrix} = \begin{bmatrix} [r_1] \\ [r_2] \\ [r_3] \\ : \\ : \\ [r_J] \end{bmatrix}$$

This block tridiagonal structure can be solved using LU method explained by Na [5].

IV. RESULTS AND DISCUSSION

The numerical results of boundary layer heat flow over a flat plate are shown in Table 1. Table 2 gives a comparison of the obtained numerical solution with HPM solution which is found to be in good agreement.

Fig.1, 2, 3 gives graphical comparison of Keller box and HPM solution for f, f' and θ .In Fig. 4 numerical results for the temperature distribution have been shown graphically for different values of the Prandtl number

Table 1 Keller Box solution of boundary layer heat flow over a flat plate when Pr=1

η	f	f'	f''	θ	θ'
0	0	0	0.332092	1	-0.33209
0.2	0.006642	0.066414	0.332016	0.933586	-0.33202
0.4	0.026562	0.132776	0.3315	0.867224	-0.3315
0.6	0.05974	0.198954	0.330107	0.801046	-0.33011
0.8	0.106116	0.26473	0.327414	0.73527	-0.32741
1	0.165583	0.329803	0.323029	0.670197	-0.32303
1.2	0.237964	0.393801	0.316608	0.606199	-0.31661
1.4	0.323	0.456288	0.307881	0.543712	-0.30788
1.6	0.420342	0.516783	0.296676	0.483217	-0.29668
1.8	0.529542	0.574784	0.282941	0.425216	-0.28294
2	0.65005	0.629791	0.266758	0.370209	-0.26676
2.2	0.78122	0.681335	0.248355	0.318665	-0.24835
2.4	0.922317	0.729005	0.228093	0.270995	-0.22809
2.6	1.072533	0.772477	0.206453	0.227523	-0.20645
2.8	1.231004	0.81153	0.184002	0.18847	-0.184
3	1.396834	0.846064	0.161354	0.153936	-0.16135
3.2	1.56912	0.8761	0.139119	0.1239	-0.13912
3.4	1.746975	0.90178	0.117865	0.09822	-0.11787
3.6	1.929549	0.923347	0.098074	0.076653	-0.09807
3.8	2.116054	0.941135	0.080112	0.058865	-0.08011
4	2.30577	0.955535	0.064219	0.044465	-0.06422
4.2	2.498064	0.966973	0.050504	0.033027	-0.0505
4.4	2.692386	0.975885	0.038957	0.024115	-0.03896
4.6	2.888274	0.982697	0.029468	0.017303	-0.02947
4.8	3.085347	0.987802	0.021857	0.012198	-0.02186
5	3.283301	0.991553	0.015893	0.008447	-0.01589

Table 2 Comparisons of results obtained by using Keller Box method and HPM [1]

η	f			f'			θ		
	HPM	NM	Keller Box	HPM	NM	Keller Box	HPM	NM	Keller Box
0	0	0	0	0	0	0	1	1	1
0.2	0.00697	0.006641	0.006642	0.0696975	0.0664077	0.0664143	0.930302	0.933592	0.933586
0.4	0.027876	0.026676	0.026562	0.1393444	0.1327641	0.1327763	0.860656	0.867236	0.867224
0.6	0.062696	0.059722	0.05974	0.2088105	0.1989372	0.198954	0.791189	0.801063	0.801046
0.8	0.111374	0.106108	0.106116	0.27788	0.2647094	0.2647296	0.72212	0.735291	0.73527
1	0.173802	0.165572	0.165583	0.3462538	0.32978	0.3298033	0.653746	0.67022	0.670197
1.2	0.249804	0.237949	0.237964	0.4135539	0.3937761	0.3938013	0.586446	0.606224	0.606199
1.4	0.339122	0.322982	0.323	0.4793309	0.4562617	0.4562879	0.520669	0.543738	0.543712
1.6	0.441401	0.420321	0.420342	0.5430747	0.5167567	0.5167832	0.456925	0.483243	0.483217
1.8	0.55618	0.529518	0.529542	0.6042289	0.5747581	0.5747843	0.395771	0.425242	0.425216
2	0.682883	0.650024	0.65005	0.6622097	0.6297657	0.6297911	0.337791	0.370234	0.370209
2.2	0.820821	0.781193	0.78122	0.7164291	0.6813103	0.6813346	0.283571	0.31869	0.318665
2.4	0.969187	0.92229	0.922317	0.7663226	0.7289819	0.729005	0.233677	0.271018	0.270995
2.6	1.127077	1.072506	1.072533	0.8113803	0.772455	0.7724769	0.18862	0.227545	0.227523
3	1.467413	1.396808	1.396834	0.8854328	0.8460444	0.8460642	0.114567	0.143955	0.153936
3.2	1.647758	1.569095	1.56912	0.914001	0.8760814	0.8761004	0.095999	0.123918	0.1239
3.4	1.83352	1.74695	1.746975	0.9369507	0.9017612	0.9017795	0.063049	0.088239	0.09822
3.6	2.023791	1.929525	1.929549	0.9545718	0.9233296	0.9233473	0.055428	0.06667	0.076653
3.8	2.217865	2.11603	2.116054	0.9673977	0.9411181	0.941135	0.032602	0.058882	0.058865
4	2.415336	2.305746	2.30577	0.9762106	0.9555182	0.9555345	0.023789	0.031482	0.044465
4.2	2.616229	2.49804	2.498064	0.9820237	0.966957	0.9669726	0.017976	0.033043	0.033027
4.4	2.821149	2.692361	2.692386	0.9860369	0.9758708	0.9758854	0.013963	0.024129	0.024115
4.6	3.031455	2.888248	2.888274	0.9895542	0.9826835	0.9826969	0.010446	0.017317	0.017303
4.8	3.249458	3.085321	3.085347	0.993854	0.9877895	0.9878018	0.006146	0.012211	0.012198
5	3.478658	3.283274	3.283301	0.9999999	0.9915419	0.991553	3.36E-10	0.008458	0.008447

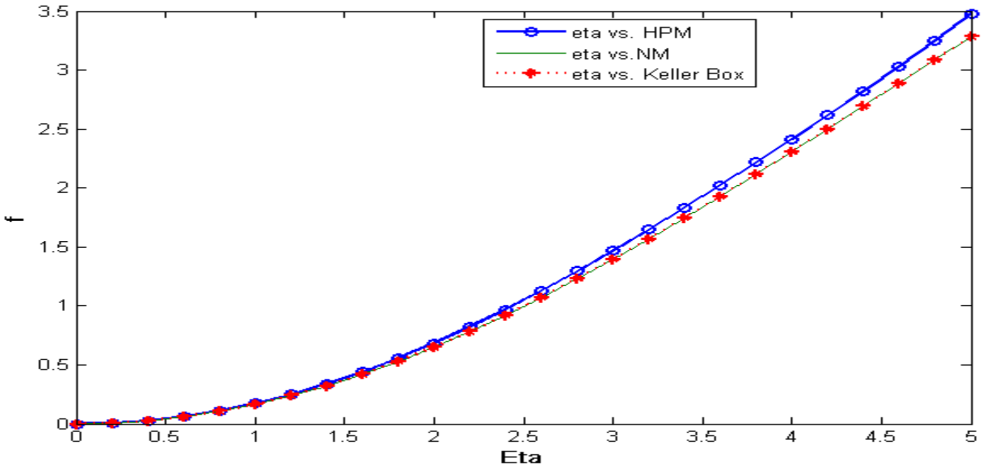

Figure 1 Comparisons of results by Keller Box method , HPM and NM for f(η)

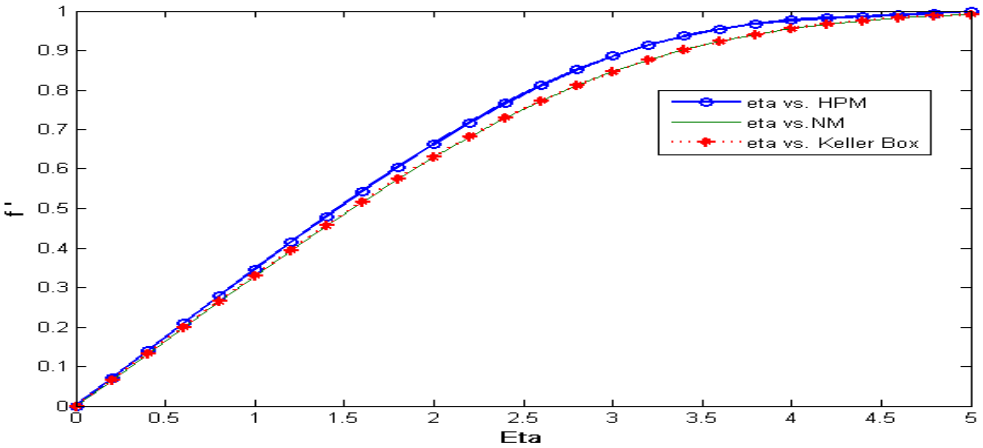

Figure 2 Comparisons of results by Keller Box method , HPM and NM for f'(η)

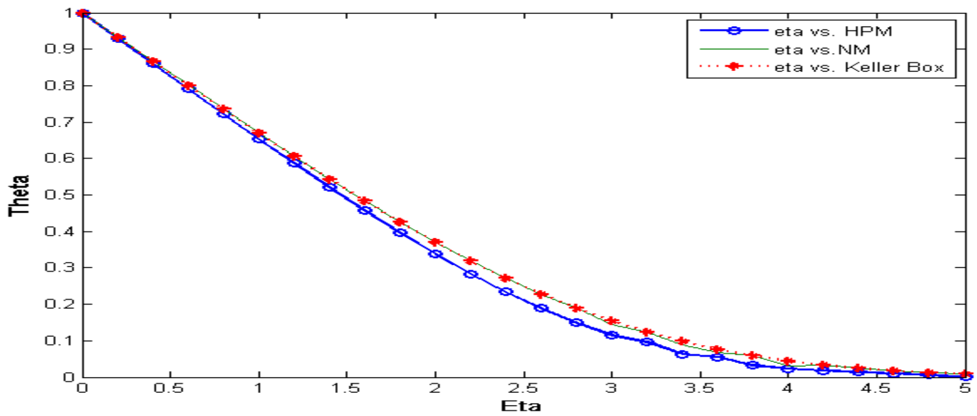

Figure 3 Comparisons of results by Keller Box method , HPM and NM for θ(η)

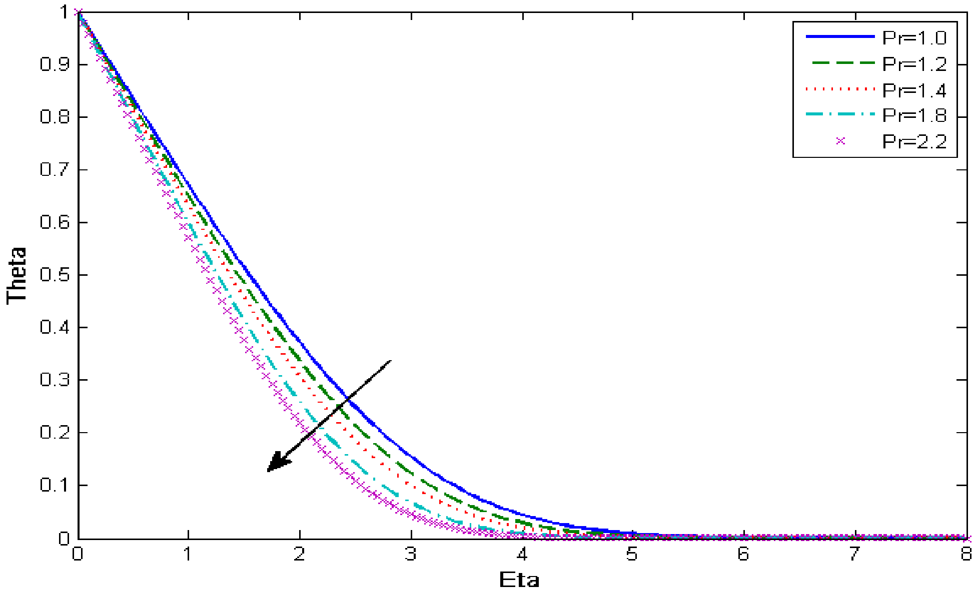

Figure 4 Effect of Prandtl Number on θ(η)

V. Conclusion

In this paper, Keller Box Method has been successfully applied to boundary layer heat transfer problem with specified boundary conditions. The obtained solutions are compared with ones from numerical method and Homotopy Perturbation Method. The excellent agreement of the Keller Box solutions and the exact solutions shows the reliability and the efficiency of the method. As prandtl number increases temperature distribution decreases which agrees well with the physical phenomena.

References

[1] M. Esmaeilpour, D.D. Ganji, Application of He's homotopy perturbation method to boundary layer flow and convection heat transfer over a flat plate, *Physics Letters A 372*, 2007, 33–38.

[2] H. Mirgolbabaei, Analytical solution of forced-convective boundary-layer flow over a flat plate, *Archives of civil and mechanical engineering, Vol. X* (2),2010, 41-51

[3] W.M. Kays, M.E. Crawford, *Convective Heat and Mass Transfer*, (3/e, McGraw–Hill, New York, 1993)

[4] Cebeci T., Bradshaw P. *Physical and computational Aspects of Convective Heat Transfer*, (New York, Springer 1988). .

[5] Na T.Y., *Computational Methods in Engineering Boundary Value Problem*, (New York: Academic Press 1979).