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Elzaki Transform Solution of One Dimensional Ground Water Recharge through Spreading

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ABSTRACT
A problem of one dimensional vertical ground water recharge has been considered by assuming the average diffusivity coefficient to be constant and conductivity as a linear function of moisture content. In the present paper we have obtained an analytical solution for the moisture content by using a new transform called ELzaki transform. We have compared the obtained analytical solution with the available Laplace transform solution which is found to be exactly same. Although the ELzaki transform have a close connection with the Laplace transform, the main advantage of ELzaki transform is that it can be used to solve problems without resorting to a new frequency domain. ELzaki transform method is easy, efficient and accurate and has got several advantages over Laplace transform method.
Keywords – conductivity, Differential Equation, Diffusivity coefficient, ELzaki transform, Moisture content.

I. INTRODUCTION

The mathematical models to the hydrological situation of one dimensional vertical ground water recharge by spreading are of great importance in water resources science, soil engineering and agricultural sciences. Many researchers have discussed this phenomenon from different aspects, for example, Klute and Hank Bower employs a finite difference method; Philips uses a transformation of variable technique; Mehta discussed multiple scale method; Verma has obtained Laplace transformation and similarity solution. In the present problem the recharge takes place over a large basin of such geological situation that the sides are limited by rigid boundaries and the bottom by a thick layer of watertable. The basic assumptions considered are the diffusivity coefficient is regarded as constant and permeability as a linear function of moisture content. The governing partial differential equation is solved by using a new transform called ELzaki transform method for a specific set of initial and boundary conditions.

II. GOVERNING EQUATION

Using the equation of continuity for an unsaturated medium and Darcy's law the governing equation [1] is given by

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right) - \frac{\rho}{\rho_s} g\,\frac{\partial K}{\partial z} \tag{1}$$

where $D = \frac{\rho K}{\rho_s}\frac{\partial\psi}{\partial\theta}$ is the diffusivity coefficient and K is the coefficient of aqueous conductivity, ρ_s the bulk density of the medium, θ is its moisture content on a dry weight basis and ρ is fluid density.

Replacing D by its average value D_a and assuming $K = K_0\theta$, we have

$$\frac{\partial\theta}{\partial t} = D_a\,\frac{\partial^2\theta}{\partial z^2} - \frac{\rho}{\rho_s}K_0\,\frac{\partial\theta}{\partial z} \tag{2}$$

Considering the watertable to be situated at a depth L, and putting

$$\frac{z}{L} = \xi,\quad \frac{tD_a}{L^2} = T$$

the resulting governing equation is given by

$$\frac{\partial\theta}{\partial T} = \frac{\partial^2\theta}{\partial\xi^2} - \frac{\rho K_0 L}{\rho_s D_a}\frac{\partial\theta}{\partial\xi} \tag{3}$$

The following initial and boundary conditions are considered

$$\theta(0,T) = \theta_0\,,\ \theta(1,T) = 1 \tag{4}$$

$$\theta(\xi,0) = 0 \tag{5}$$

where the moisture content throughout the region is zero initially, at the layer $z = 0$ it is θ_0 and at the water table $z = L$ it is assumed to remain 100% throughout the process of investigation. It may be remarked that the effect of capillary action at the stationary groundwater level being small is neglected.

III. ELZAKI TRANSFORM

A new transform called the ELzaki transform defined for function of exponential order we consider functions in the set A defined by: [2]

$$A = \left\{f(t): \exists M\,, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, \text{if } t \in (-1)^j \times [0,\infty)\right\} \tag{6}$$

For a given function in the set A, the constant M must be finite number, k_1, k_2 may be finite or infinite. Tarig Elzaki introduced a new transform and named as ELzaki transform which is defined by the integral equation

$$E[f(t)] = \hat{T}(v) = v\int_0^{\infty} f(t)e^{-\frac{t}{v}}dt \ , \quad t \geq 0\,, k_1 \leq v \leq k_2 \qquad (7)$$

ELzaki transform of some functions

$$E[1] = \hat{T}(v) = v\int_0^{\infty} 1\ e^{-\frac{t}{v}}\,dt = v\left[-ve^{-\frac{t}{v}}\right]_0^{\infty} = v^2;$$

$$E[e^{at}] = \hat{T}(v) = v\int_0^{\infty} e^{at}\ e^{-\frac{t}{v}}\,dt = \frac{v^2}{1-av}\ ; E[e^{-at}] = \hat{T}(v) = v\int_0^{\infty} e^{-at}\ e^{-\frac{t}{v}}\,dt = \frac{v^2}{1+av}\ ;$$

$$E[\sin at] = \frac{av^3}{1+a^2v^2}\ ;\ E[\cos at] = \frac{v^2}{1+a^2v^2}\ ;$$

$$E[\sinh at] = \frac{av^3}{1-a^2v^2}\ ;\ E[\cosh at] = \frac{v^2}{1-a^2v^2}.$$

Theorem: If $E[f(t)] = \hat{T}(v)$ then [2]

(i) $E\left[\frac{df}{dt}\right] = E[f'(t)] = \frac{\hat{T}(v)}{v} - v\,f(0)$

(ii) $E\left[\frac{d^2f}{dt^2}\right] = E[f''(t)] = \frac{\hat{T}(v)}{v^2} - f(0) - v\,f'(0)$

(iii) $E\left[\frac{d^nf}{dt^n}\right] = E[f^n(t)] = \frac{\hat{T}(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k}\ f^{(k)}(0)$

IV. ANALYTICAL SOLUTION

Setting $\frac{\rho K_0 L}{\rho_s D_a} = \beta$ in equation (3), we get

$$\frac{\partial\theta}{\partial T} = \frac{\partial^2\theta}{\partial\xi^2} - \beta\frac{\partial\theta}{\partial\xi} \qquad (8)$$

with $\theta(0,T) = \theta_0\,,\ \theta(1,T) = 1$ and $\theta(\xi,0) = 0$

Let $\hat{T}(\xi,v)$ be the ELzaki transform of $\theta(\xi,T)$ i.e. $E[\theta(\xi,T)] = \hat{T}(\xi,v)$. Then taking the ELzaki transform on both side of equation (8) we have:

$$\frac{\hat{T}(\xi,v)}{v} - v\,\theta(\xi,0) = \frac{d^2\hat{T}}{d\xi^2} - \beta\frac{d\hat{T}}{d\xi}$$

Using condition (5) we obtain

$$\frac{d^2\hat{T}}{d\xi^2} - \beta\frac{d\hat{T}}{d\xi} - \frac{\hat{T}}{v} = 0 \qquad (9)$$

where $\hat{T}(\xi,v) = v\int_0^{\infty} e^{\frac{-t}{v}}\,\theta(\xi,T)\,dt$ [3] represents the ELzaki transform of $\theta(\xi,T)$.

The ELzaki transform of the boundary conditions (4) yields

$$\hat{T}(0,v) = v^2\theta_0 \quad , \quad \hat{T}(1,v) = v^2 \qquad (10)$$

Since the equation (9) is a linear differential equation with constant coefficient, we may write its general solution as:

$$\hat{T}(\xi,v) = \left\{A\cosh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right) + B\sinh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right)\right\}e^{\frac{\beta}{2}\xi} \qquad (11)$$

where A and B are constants of integration. For evaluating A and B, we apply conditions (10) to equation (11), so that after some simplification, we have

$$A = v^2\theta_0 \quad \text{and} \quad B = \frac{v^2e^{\frac{-\beta}{2}} - v^2\theta_0\cosh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}}{\sinh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}}$$

Substituting these values in equation (11), we have:

$$\hat{T}(\xi,v) = \left\{v^2\theta_0\cosh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right) + \frac{v^2e^{\frac{-\beta}{2}} - v^2\theta_0\cosh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}}{\sinh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}}\sinh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right)\right\}e^{\frac{\beta}{2}\xi}$$

$$= e^{\frac{\beta}{2}\xi}v^2\theta_0\cosh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right) + \frac{v^2e^{\frac{-\beta}{2}}e^{\frac{\beta}{2}\xi}}{\sinh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}}\sinh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right) - \frac{v^2\theta_0e^{\frac{\beta}{2}\xi}\cosh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}}{\sinh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}}\sinh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right)$$

$$\hat{T}(\xi,v) = \frac{e^{\frac{\beta}{2}\xi}v^2\theta_0\sinh\left((1-\xi)\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\right)}{\sinh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}} + \frac{e^{\frac{-\beta}{2}(1-\xi)}v^2\sinh\left(\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}\ \xi\right)}{\sinh\sqrt{\frac{\beta^2}{4}+\frac{1}{v}}} \qquad (12)$$

The inverse transform (E^{-1}) of the right hand side terms in equation (12) may be determined by recalling a standard result as [4, 5]

$E^{-1}\left[\hat{T}(s)\right] = \sum$ residues of $\left[e^{st}s\,\hat{T}\left(\frac{1}{s}\right)\right]$, where $\hat{T}(v)$ is transform of f(t). Therefore,

$$E^{-1}\left[\hat{T}(\xi,v)\right] = \sum \text{residues of } \left[e^{sT}s\,\hat{T}\left(\xi,\frac{1}{s}\right)\right] \qquad (13)$$

Consider the right hand side terms of equation (12) as

$$\hat{T}_1\left(\xi,\frac{1}{s}\right) = \frac{\sinh\left((1-\xi)\sqrt{\frac{\beta^2}{4}+s}\right)}{s^2\ \sinh\sqrt{\frac{\beta^2}{4}+s}} \text{ and}$$

$$\hat{T}_2\left(\xi,\frac{1}{s}\right) = \frac{\sinh\left(\xi\sqrt{\frac{\beta^2}{4}+s}\right)}{s^2\ \sinh\sqrt{\frac{\beta^2}{4}+s}}$$

Therefore inverse transform of the right hand terms in (12) is

$$E^{-1}\left[\hat{T}(\xi,v)\right] = e^{\frac{\beta}{2}\xi}\theta_0\sum \text{residues of } \left[e^{sT}s\,\hat{T}_1\left(\xi,\frac{1}{s}\right)\right] + e^{\frac{-\beta}{2}(1-\xi)}\sum \text{residues of } \left[e^{sT}s\,\hat{T}_2\left(\xi,\frac{1}{s}\right)\right] \qquad (14)$$

Noting that $\hat{T}_1\left(\xi,\frac{1}{s}\right)$ and $\hat{T}_2\left(\xi,\frac{1}{s}\right)$ has poles at $s = 0$(double pole) and other pole at $s = -n^2\pi^2 - \frac{\beta^2}{4}$.

Now we calculate residues one by one

Residue of $e^{sT} s\, \widehat{T}_1\left(\xi, \frac{1}{s}\right)$ at $(s = 0)$

$$= \lim_{s \to 0} \frac{d}{ds}\left[(s-0)^2\, e^{sT} s \frac{\sinh\left((1-\xi)\sqrt{\frac{\beta^2}{4}+s}\right)}{s^2\, \sinh\sqrt{\frac{\beta^2}{4}+s}}\right]$$

$$= \frac{\sinh\,(1-\xi)\frac{\beta}{2}}{\sinh\frac{\beta}{2}}$$

Residue of $e^{sT} s\, \widehat{T}_1\left(\xi, \frac{1}{s}\right)$ at $\left(s = -n^2\pi^2 - \frac{\beta^2}{4}\right) = \frac{-2n\pi\, \sin\xi n\pi}{n^2\pi^2 + \frac{\beta^2}{4}} e^{-\left(n^2\pi^2 + \frac{\beta^2}{4}\right)T}$

Similarly,

Residue of $e^{sT} s\, \widehat{T}_2\left(\xi, \frac{1}{s}\right)$ at $(s = 0) = \frac{\sinh\ \xi\frac{\beta}{2}}{sinh\frac{\beta}{2}}$ and

Residue of $e^{sT} s\, \widehat{T}_2\left(\xi, \frac{1}{s}\right)$ at $\left(s = -n^2\pi^2 - \frac{\beta^2}{4}\right) = \frac{2n\pi\,(-1)^n \sin\xi n\pi}{n^2\pi^2 + \frac{\beta^2}{4}} e^{-\left(n^2\pi^2 + \frac{\beta^2}{4}\right)T}$

Putting all these values of residues in equation (14) yields:

$$\begin{aligned}\theta(\xi, T) &= e^{\frac{\beta}{2}\xi}\theta_0 \left\{\frac{\sinh\,(1-\xi)\frac{\beta}{2}}{\sinh\frac{\beta}{2}} - 2\pi \sum_{n=1}^{\infty} \frac{n\, \sin\xi n\pi}{n^2\pi^2 + \frac{\beta^2}{4}} e^{-\left(n^2\pi^2 + \frac{\beta^2}{4}\right)T}\right\} \\ &+ e^{\frac{-\beta}{2}(1-\xi)} \left\{\frac{\sinh\ \xi\frac{\beta}{2}}{\sinh\frac{\beta}{2}} + 2\pi \sum_{n=1}^{\infty} \frac{2n\pi\,(-1)^n \sin\xi n\pi}{n^2\pi^2 + \frac{\beta^2}{4}} e^{-\left(n^2\pi^2 + \frac{\beta^2}{4}\right)T}\right\} \qquad (15)\end{aligned}$$

This is the desired analytical expression for the moisture content distribution which is same as the Laplace transform solution [1].

V. RESULT AND DISCUSSION

The following values of the parameters are considered. $\theta_0 = 0.1, \beta = 0.4$

The numerical values for the moisture content with the above set of values for different times are shown in Table 1. The graphical representation of the moisture content for various values of time is shown in Fig.1. From Fig. 1 it is observed that the moisture content increases with time and the space co-ordinate which matches well with phenomena. The obtained analytical solution is same as the available Laplace transform solution. However, in the Laplace transform method the frequency domain is resorted while in the ELzaki transform method this difficulty is overcome. The ELzaki transform may be used in several such engineering problems without resorting to a new frequency domain.

Table 1: moisture content at different time

Xi	T=0.1	T=0.2	T=0.3	T=0.4	T=0.6	T=1
0	0.1	0.1	0.1	0.1	0.1	0.1
0.1	0.10912	0.14942	0.16529	0.17119	0.1745	0.17467
0.2	0.12444	0.20338	0.23418	0.24564	0.25206	0.25239
0.3	0.15197	0.26444	0.30771	0.32381	0.33282	0.33328
0.4	0.19754	0.33486	0.38681	0.40611	0.41693	0.41748
0.5	0.26619	0.41643	0.47222	0.49292	0.50452	0.50512
0.6	0.36159	0.51023	0.56441	0.5845	0.59576	0.59633
0.7	0.48525	0.61651	0.66357	0.68101	0.69078	0.69127
.8	0.63602	0.73465	0.76956	0.78248	0.78972	0.79009
0.9	0.80984	0.8632	0.88193	0.88887	0.89275	0.89295
1	1	1	1	1	1	1

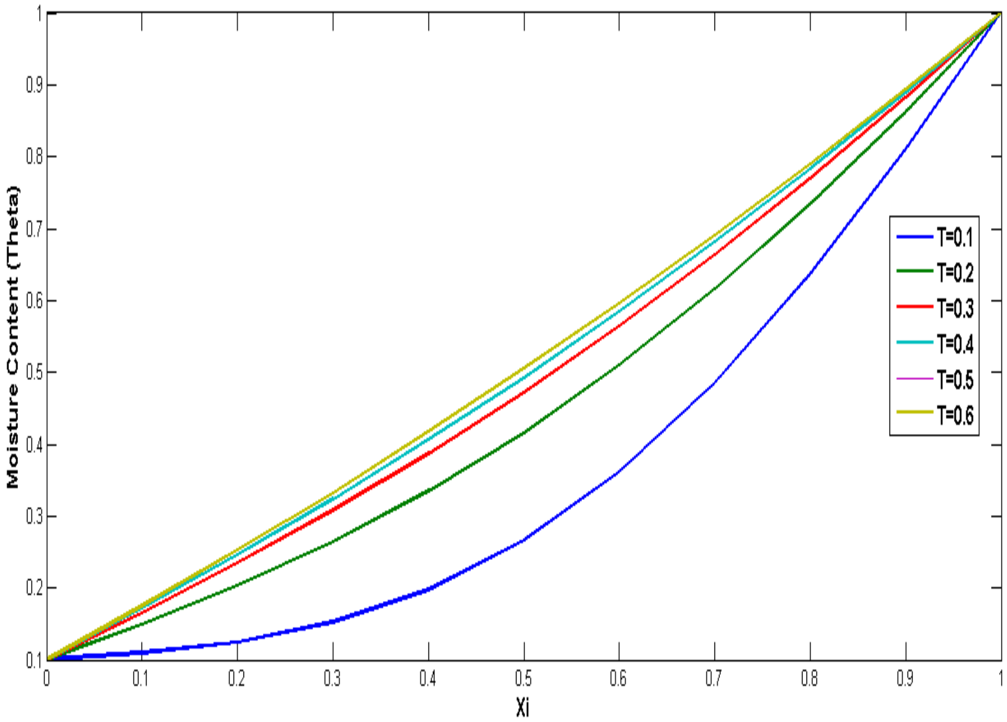

Figure 1: Representing moisture content at different time

REFERENCES

[1] A. P. Verma, The Laplace transform solution of a one dimensional groundwater recharge by spreading, Annali Di Geofision, 22(1), 1969, 25-31.

[2] Tarig M Elzaki, The new integral transform "ELzaki Transform", Global Journal of Pure and Applied Mathematics, 7(1), 2011, 57-64.

[3] Tarig M. Elzaki and Salih M. Elzaki, Application of new transform "ELzaki Transform" to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, 7(1), 2011, 65-70.

[4] Tarig M. Elzaki and Salih M. Elzaki, On the connection between Laplace and ELzaki transforms, Advances in Theoretical and Applied Mathematics, 6(1), 2011, 1-10.

[5] Tarig M. Elzaki, Salih M. Elzaki and Elsayed A. Elnour, On the new integral transform "ELzaki Transform" fundamental properties investigation and applications, 4(1), 2012, 1-13.