Multi - Anti Fuzzy Group And Its Lower Level Subgroups

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Abstract
In this paper, we define the algebraic structures of multi-anti fuzzy subgroup and some related properties are investigated. The purpose of this study is to implement the fuzzy set theory and group theory in multi-anti fuzzy subgroups. Characterizations of multi-lower level subsets of a multi-anti fuzzy subgroup of a group are given.

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I. Introduction

II. PRELIMINARIES
In this section, we site the fundamental definitions that will be used in the sequel.

2.1 Definition
Let X be any non-empty set. A fuzzy subset μ of X is μ : X → [0, 1].

2.2 Definition
Let X be a non-empty set. A multi-fuzzy set A in X is defined as a set of ordered sequences:

\[ A = \{ (x, \mu_1(x), \mu_2(x), ..., \mu_k(x)) : x \in X \} \]

where \( \mu_i : X \rightarrow [0, 1] \) for all i.

Remark
i. If the sequences of the membership functions have only k-terms (finite number of terms), k is called the dimension of A.
ii. The set of all multi-fuzzy sets in X of dimension k is denoted by \( \mathcal{M}^k \text{FS}(X) \).
iii. The multi-fuzzy membership function \( \mu_A \) is a function from X to \([0, 1]^k\) such that for all x in X, \( \mu_A(x) = (\mu_1(x), \mu_2(x), ..., \mu_k(x)) \).
iv. For the sake of simplicity, we denote the multi-fuzzy set

\[ A = \{ (x, \mu_1(x), \mu_2(x), ..., \mu_k(x)) : x \in X \} \]

as \( A = (\mu_1, \mu_2, ..., \mu_k) \).

2.3 Definition
Let k be a positive integer and let A and B in \( \mathcal{M}^k \text{FS}(X) \), where \( A = (\mu_1, \mu_2, ..., \mu_k) \) and \( B = (v_1, v_2, ..., v_k) \), then we have the following relations and operations:

i. \( A \subseteq B \) if and only if \( \mu_i \leq v_i \), for all i = 1, 2, ..., k;
ii. \( A = B \) if and only if \( \mu_i = v_i \), for all i = 1, 2, ..., k;
iii. \( A \cup B = (\mu_1 \cup v_1, ..., \mu_k \cup v_k) = \{ (x, \max(\mu_1(x), v_1(x)), ..., \max(\mu_k(x), v_k(x))) : x \in X \} \);
iv. \( A \cap B = (\mu_1 \cap v_1, ..., \mu_k \cap v_k) = \{ (x, \min(\mu_1(x), v_1(x)), ..., \min(\mu_k(x), v_k(x))) : x \in X \} \);
v. \( A + B = (\mu_1 + v_1, ..., \mu_k + v_k) = \{ (x, \mu_1(x) + v_1(x), ..., \mu_k(x) + v_k(x)) : x \in X \} \).

2.4 Definition
Let \( A = (\mu_1, \mu_2, ..., \mu_k) \) be a multi-fuzzy set of dimension k and let \( \mu'_i \) be the fuzzy complement of the ordinary fuzzy set \( \mu_i \) for i = 1, 2, ..., k. The Multi-fuzzy Complement of the multi-fuzzy set A is a multi-fuzzy set \( (\mu'_1, ..., \mu'_k) \) and it is denoted by \( C(A) \) or \( A^c \).

That is, \( C(A) = \{ (x, c(\mu_1(x)), ..., c(\mu_k(x))) : x \in X \} = \{ (x, 1-\mu_1(x), ..., 1-\mu_k(x)) : x \in X \} \), where c is the fuzzy complement operation.

2.5 Definition
Let A be a fuzzy set on a group G. Then A is said to be a fuzzy subgroup of G if for all \( x, y \in G \),

i. \( A(xy) \geq \min \{ A(x), A(y) \} \)
ii. \( A(x^{-1}) = A(x) \).
2.6 Definition
A multi-fuzzy set A of a group G is called a multi-fuzzy subgroup of G if for all x, y ∈ G,
i. A(xy) ≤ min{A(x), A(y)}
ii. A(x⁻¹) = A(x)

2.7 Definition
Let A be a fuzzy set on a group G. Then A is called an anti fuzzy subgroup of G if for all x, y ∈ G,
i. A(xy) ≤ max{A(x), A(y)}
ii. A(x⁻¹) = A(x)

2.8 Definition
A multi-fuzzy set A of a group G is called a multi-anti fuzzy subgroup of G if for all x, y ∈ G,
i. A(xy) ≤ max{A(x), A(y)}
ii. A(x⁻¹) = A(x)

2.9 Definition
Let A and B be any two multi-fuzzy sets of a non-empty set X. Then for all x ∈ X,
i. A ⊆ B iff A(x) ≤ B(x),
ii. A = B iff A(x) = B(x),
iii. A ∪ B(x) = max{A(x), B(x)},
iv. A ∩ B(x) = min{A(x), B(x)}.

3.0 Definition
Let A and B be any two multi-fuzzy sets of a non-empty set X. Then
i. A ∪ A = A, A ∩ A = A,
ii. A ⊆ A ∪ B, B ⊆ A ∪ B, A ∩ A ⊆ A and A ∩ B ⊆ B,
iii. A ⊆ B iff A ∪ B = B,
iv. A ⊆ B iff A ∩ B = A.

III. Properties of multi-anti fuzzy subgroups
In this section, we discuss some of the properties of multi-anti fuzzy subgroups.

3.1 Theorem
Let ‘A’ be a multi-anti fuzzy subgroup of a group G and ‘e’ is the identity element of G. Then
i. A(x) ≥ A(e) for all x ∈ G,
ii. The subset H = {x ∈ G / A(x) = A(e)} is a subgroup of G.

Proof
i. Let x ∈ G.
\[ A(x) = \text{max} \{ A(x), A(e) \} \]
\[ = \text{max} \{ A(x), A(x⁻¹) \} \]
\[ ≥ A(x⁻¹) \]
\[ = A(e). \]
Therefore, A(x) ≥ A(e), for all x ∈ G.
ii. Let H = {x ∈ G / A(x) = A(e)}
Clearly H is non-empty as e ∈ H.
Let x, y ∈ H. Then, A(x) = A(y) = A(e)
\[ A(xy^{-1}) ≤ \text{max} \{ A(x), A(y^{-1}) \} \]
\[ = \text{max} \{ A(x), A(y) \} \]
\[ = \text{max} \{ A(e), A(e) \} \]
\[ = A(e). \]
That is, A(xy⁻¹) ≤ A(e) and obviously A(xy⁻¹) ≥ A(e) by i.
Hence, A(xy⁻¹) = A(e) and xy⁻¹ ∈ H.
Clearly, H is a subgroup of G.

3.2 Theorem
A is a multi-fuzzy subgroup of G iff A is a multi-anti fuzzy subgroup of G.

Proof
Suppose A is a multi-fuzzy subgroup of G.
Then for all x, y ∈ G,
\[ A(xy) ≥ \text{min} \{ A(x), A(y) \} \]
\[ ⇔ 1 − A′(xy) ≥ \text{min} \{ 1 − A′(x), (1 − A′(y)) \} \]
\[ ⇔ A′(xy) ≤ 1 − \text{min} \{ 1 − A′(x), (1 − A′(y)) \} \]
\[ ⇔ A′(xy) ≤ \text{max} \{ A′(x), A′(y) \} \].
We have, A′(x) = A(x⁻¹) for all x in G
\[ ⇔ 1 − A′(x) = 1 − A(x⁻¹) \]
Therefore
\[ A′(x) = A′(x⁻¹) \].
Hence A′ is a multi-anti fuzzy subgroup of G.

3.3 Theorem
Let A be any multi-anti fuzzy subgroup of a group G with identity ‘e’. Then A(xy⁻¹) = A(e) ⇒ A(x) = A(y) for all x, y in G.

Proof
Given A is a multi-anti fuzzy subgroup of G and A(xy⁻¹) = A(e).
Then for all x, y in G,
\[ A(x) = A(xy⁻¹y) \]
\[ = A(xy⁻¹) \]
\[ ≤ \text{max} \{ A(xy⁻¹), A(y) \} \]
\[ = \text{max} \{ A(e), A(y) \} \]
\[ = A(y). \]
That is, A(y) ≤ A(x).
Now, A(y) = A(y⁻¹), since A is a multi-anti fuzzy subgroup of G.
\[ = A(ey⁻¹) \]
\[ = A((x⁻¹x)y⁻¹) \]
\[ = A((x⁻¹(x⁻¹))y⁻¹) \]
\[ ≤ \text{max} \{ A(x⁻¹), A(xy⁻¹) \} \]
\[ = \text{max} \{ A(x), A(e) \} \]
\[ = A(x). \]
That is, A(y) ≤ A(x).
Hence, A(x) = A(y).

3.4 Theorem
A is a multi-anti fuzzy subgroup of a group G
\[ ⇔ A(xy⁻¹) ≤ \text{max} \{ A(x), A(y) \} \]
for all x, y in G.

Proof
Let A be a multi-anti fuzzy subgroup of a group G.
Then for all x, y in G,
\[ A(xy) ≤ \text{max} \{ A(x), A(y) \} \]
and
\[ A(x) = A(x⁻¹) \].
Now, A(xy⁻¹) ≤ \text{max} \{ A(x), A(y⁻¹) \}
\[ = \text{max} \{ A(x), A(y) \} \]
\[ ⇔ A(xy⁻¹) ≤ \text{max} \{ A(x), A(y) \}. \]

IV. Properties of multi-lower level subsets of a multi-anti fuzzy subgroup
In this section, we introduce the concept of multi-lower level subset of a multi-anti fuzzy subgroup and discuss some of its properties.

4.1 Definition
Let A be a multi-anti fuzzy subgroup of a group G. For any \( t = (t_1, t_2, \ldots, t_k, \ldots) \) where \( t_i \in [0,1] \) for all \( i \), we define the multi-lower level subset of A is the set, \( L(A; t) = \{ x \in G / A(x) \leq t \} \).

4.1 Theorem
Let A be a multi-anti fuzzy subgroup of a group G. Then for any \( t = (t_1, t_2, \ldots, t_k, \ldots) \), where \( t_i \in [0,1] \) for all \( i \) such that \( t \geq A(e) \), where ‘e’ is the identity element of G, \( L(A; t) \) is a subgroup of G.

Proof
For all \( x, y \in L(A; t) \), we have,
\[
A(x) \leq t ; \quad A(y) \leq t.
\]
Now,
\[
A(xy^{-1}) \leq \max \{A(x), A(y)\} \leq \max \{t, t\} = t.
\]
That is, \( A(xy^{-1}) \leq t \)
Therefore, \( xy^{-1} \in L(A; t) \).
Hence \( L(A; t) \) is a subgroup of G.

4.2 Theorem
Let G be a group and A be a multi-fuzzy subset of G such that \( L(A; t) \) is a subgroup of G. Then for \( t = (t_1, t_2, \ldots, t_k, \ldots) \), where \( t_i \in [0,1] \) for all \( i \) such that \( t \geq A(e) \) where ‘e’ is the identity element of G, A is a multi-anti fuzzy subgroup of G.

Proof
Let \( x, y \in G \) and \( A(x) = r \) and \( A(y) = s \), where \( r = (r_1, r_2, \ldots, r_k, \ldots) \), \( s = (s_1, s_2, \ldots, s_k, \ldots) \), \( r_i, s_i \in [0,1] \) for all \( i \).
Suppose \( r \leq s \).
Now \( A(y) = s \) which implies \( y \in L(A; s) \).
And now \( A(x) = r < s \) which implies \( x \in L(A; s) \).
Therefore, \( x, y \in L(A; s) \).
As \( L(A; s) \) is a subgroup of G,
\[
xy^{-1} \in L(A; s).
\]
Hence,
\[
A(xy^{-1}) \leq s = \max\{r, s\} \leq \max\{A(x), A(y)\}.
\]
That is, \( A(xy^{-1}) \leq \max\{A(x), A(y)\} \).
Hence A is a multi-anti fuzzy subgroup of G.

4.2 Definition
Let A be a multi-anti fuzzy subgroup of a group G. The subgroups \( L(A; t) \) for \( t = (t_1, t_2, \ldots, t_k, \ldots) \) where \( t_i \in [0,1] \) for all \( i \) and \( t \geq A(e) \) where ‘e’ is the identity element of G, are called multi-lower level subgroups of A.

4.3 Theorem
Let A be a multi-anti fuzzy subgroup of a group G and ‘e’ is the identity element of G. If two multi-lower level subgroups \( L(A; r) \), \( L(A; s) \), for \( r = (r_1, r_2, \ldots, r_k, \ldots) \), \( s = (s_1, s_2, \ldots, s_k, \ldots) \), \( r_i, s_i \in [0,1] \) for all \( i \) and \( r, s \geq A(e) \) with \( r < s \) of A are equal, then there is no x in G such that \( r < A(x) \leq s \).

Proof
Let \( L(A; r) = L(A; s) \).
Suppose there exists \( a \in G \) such that \( r < A(x) \leq s \).
Then \( L(A; r) \subseteq L(A; s) \).
That is, \( x \in L(A; s) \), but \( x \notin L(A; r) \), which contradicts the assumption that \( L(A; r) = L(A; s) \).
Hence there is no x in G such that \( r < A(x) \leq s \).

Conversely, Suppose there is no x in G such that \( r < A(x) \leq s \).
Then, by definition, \( L(A; r) \subseteq L(A; s) \).
Let \( x \in L(A; s) \) and there is no x in G such that \( r < A(x) \leq s \).
Hence \( L(A; r) = L(A; s) \).

4.4 Theorem
A multi-fuzzy subset A of G is a multi-anti fuzzy subgroup of a group G if and only if the multi-lower level subgroups \( L(A; t) \), for \( t = (t_1, t_2, \ldots, t_k, \ldots) \) where \( t_i \in [0,1] \) for all \( i \) and \( t \geq A(e) \), are subgroups of G.

Proof
It is clear.

4.5 Theorem
Any subgroup H of a group G can be realized as a multi-lower level subgroup of some multi-anti fuzzy subgroup of G.

Proof
Let A be a multi-anti fuzzy subset and \( x \in G \).
Define,
\[
A(x) = \begin{cases} 
0 & \text{if } x \in H \\
\min(t, s) & \text{if } x \notin H, \text{ for } t = (t_1, t_2, \ldots, t_k, \ldots) \text{ such that } t_i \in [0,1] \text{ for all } i \text{ and } t \geq A(e) \text{ in } A \text{ and } A(y) = s \text{ in } A \text{ and } A(z) = t \text{ in } A.
\end{cases}
\]
where \( t_i \in [0,1] \) for all \( i \).
we shall prove that A is a multi-anti fuzzy subgroup of G. Let \( x, y \in G \).

i. Suppose \( x, y \notin H \). Then \( xy \notin H \) and \( x^{-1} \notin H \).
\( A(x) = 0 \), \( A(y) = 0 \), \( A(xy^{-1}) = 0 \).
Hence \( A(xy^{-1}) \leq \max\{A(x), A(y)\} \).

ii. Suppose \( x \notin H \) and \( y \in H \). Then \( xy \notin H \) and \( x^{-1} \in H \).
\( A(x) = 0 \), \( A(y) = t \), \( A(xy^{-1}) = t \).
Hence \( A(xy^{-1}) = \min\{A(x), A(y)\} \).

iii. Suppose \( x \in H \) and \( y \notin H \). Then \( xy \notin H \) and \( x^{-1} \notin H \).
\( A(x) = t \), \( A(y) = 0 \), \( A(xy^{-1}) = 0 \).
Hence \( A(xy^{-1}) = \max\{A(x), A(y)\} \).
Thus in all cases, A is a multi-anti fuzzy subgroup of G. For this multi-anti fuzzy subgroup A, \( L(A; t) = H \).

Remark
As a consequence of the Theorem 4.3, the multi-lower level subgroups of a multi-anti fuzzy subgroup A of a group G form a chain. Since \( A(e) \leq A(x) \) for all \( x \in G \) where ‘e’ is the identity element of G, therefore \( L(A; t_0) \), where \( A(e) = t_0 \) is the smallest and we have the chain :
\[
[e] \subseteq L(A; t_0) \subseteq L(A; t_1) \subseteq L(A; t_2) \subseteq \ldots \subseteq L(A; t_n) = G
\]
References


