Homomorphism and Anti Homomorphism on a Bipolar Anti Fuzzy Subgroup

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ABSTRACT
In this paper, we introduce the concept of an image, anti pre-image of a bipolar fuzzy subgroup of a group G and discuss in detail a series of homomorphic and anti homomorphic properties of bipolar fuzzy and bipolar anti fuzzy subgroup.

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I. Introduction
The concept of fuzzy sets was initiated by Zadeh [16]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [12] gave the idea of fuzzy subgroup. In fuzzy sets the membership degree of elements range over the interval [0,1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0,1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. The author W.R.Zhang [14],[15] commenced the concept of bipolar fuzzy subsets as a generalization of fuzzy sets in 1994. In case of bipolar-valued fuzzy subsets membership degree range is enlarged from the interval [0, 1] to [-1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property and the membership degree [-1,0) indicates that elements somewhat satisfy the implicit counter-property. M. Marudai, V.Rajendran [5] introduced the pre-image of bipolar Fuzzy subgroup. In this paper we redefined the concept of a pre-image of a bipolar fuzzy subgroup and introduce the concept of an image, anti image and anti pre-image of a bipolar fuzzy subgroup and discuss some of its properties with bipolar anti fuzzy subgroup.

II. Preliminaries
In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G , *) be a finite group, e is the identity element of G, xy we mean x * y.

Definition 2.1 [1]
Let X be any non-empty set. A fuzzy subset μ of X is a function μ : X → [0,1].

Definition 2.2 [9]
Let G be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set μ in G is an object having the form μ = {(x, μ⁺(x), μ⁻(x)) / for all x ∈ G}, where μ⁺ : G → [0,1] and μ⁻ : G → [-1,0] are mappings. The positive membership degree μ⁺(x) denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set μ = {(x, μ⁺(x), μ⁻(x)) / for all x ∈ G} and the negative membership degree μ⁻(x) denotes the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar-valued fuzzy set μ = {(x, μ⁺(x), μ⁻(x)) / for all x ∈ G}. If μ⁺(x) ≠ 0 and μ⁻(x) = 0, it is the situation that x is regarded as having only positive satisfaction for μ = {(x, μ⁺(x), μ⁻(x)) / for all x ∈ G}. If μ⁻(x) = 0 and μ⁺(x) ≠ 0, it is the situation that x does not satisfy the property of μ = {(x, μ⁺(x), μ⁻(x)) / for all x ∈ G}, but somewhat satisfies the counter property of μ = {(x, μ⁻(x), μ⁺(x)) / for all x ∈ G}. It is possible for an element x to be such that μ⁺(x) ≠ 0 and μ⁻(x) ≠ 0 when the membership function of property overlaps that its counter property over some portion of G. For the sake of simplicity, we shall use the symbol μ = (μ⁺, μ⁻) for...
the bipolar-valued fuzzy set $\mu = \{(x, \mu^+(x), \mu^-(x))\}$ for all $x \in G$.

**Definition 2.3[9]**

Let $G$ be a group. A bipolar-valued fuzzy set or bipolar fuzzy set $\mu$ in $G$ is a bipolar fuzzy subgroup of $G$ if for all $x, y \in G$,

1. $\mu^+(xy) \geq \min\{\mu^+(x), \mu^+(y)\}$,
2. $\mu^-(xy) \leq \max\{\mu^-(x), \mu^-(y)\}$,
3. $\mu^+(x^{-1}) = \mu^+(x), \mu^-(x^{-1}) = \mu^-(x)$.

**Definition 2.4[9]**

Let $G$ be a group. A bipolar-valued fuzzy set or bipolar fuzzy set $\mu$ in $G$ is a bipolar anti fuzzy subgroup of $G$ if for all $x, y \in G$,

1. $\mu^+(xy) \geq \min\{\mu^+(x), \mu^+(y)\}$,
2. $\mu^-(xy) \leq \max\{\mu^-(x), \mu^-(y)\}$,
3. $\mu^+(x^{-1}) = \mu^+(x), \mu^-(x^{-1}) = \mu^-(x)$.

**Definition 2.5[7]**

A mapping $f$ from a group $G_1$ to a group $G_2$ is said to be a homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in G_1$.

**Definition 2.6[7]**

A mapping $f$ from a group $G_1$ to a group $G_2$ ($G_1$ and $G_2$ are not necessarily commutative) is said to be an anti homomorphism if $f(xy) = f(y)f(x)$ for all $x, y \in G_1$.

**Theorem 2.1**

Let $\mu$ be a bipolar fuzzy set of $G$, then $\mu$ is a bipolar anti fuzzy subgroup of $G$ if and only if $\mu^-$ is a bipolar fuzzy subgroup of $G$.

**Proof**

Let $\mu = (\mu^+, \mu^-)$ be a bipolar anti fuzzy subgroup of $G$. Then for each $x, y \in G$,

Now

1. $\mu^+(xy) \leq \max\{\mu^+(x), \mu^+(y)\}$
2. $\mu^-(xy) \leq \min\{\mu^-(x), \mu^-(y)\}$
3. $\mu^+(x^{-1}) = \mu^+(x), \mu^-(x^{-1}) = \mu^-(x)$

Hence $\mu^-(x) = (\mu^+(x), \mu^-(x))$ is a bipolar fuzzy subgroup of $G$.

**Definition 2.7[10]**

Let $f$ be a mapping from a group $G_1$ to a group $G_2$. Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ be bipolar fuzzy subsets of $G_1$ and $G_2$ respectively, then the image of $\mu$ under $f$ is a bipolar fuzzy subset $\mu^f = (f(\mu)^+, f(\mu)^-)$ of $G_2$ defined by for each $u \in G$,

$$f(\mu)^+(u) = \begin{cases} \max\{\mu^+(x): x \in f^{-1}(u)\} & \text{if } f^{-1}(u) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(\mu)^-(u) = \begin{cases} \max\{\mu^-(x): x \in f^{-1}(u)\} & \text{if } f^{-1}(u) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

also the pre-image $f^{-1}(\varphi)$ of $\varphi$ under $f$ is a bipolar fuzzy subset of $G_1$ defined by for $x \in G_1$, $f^{-1}(\varphi)^+(x) = \varphi^+(f(x)), f^{-1}(\varphi)^-(x) = \varphi^-(f(x))$.

**III. Properties of a Bipolar Anti Fuzzy Group of a Group under Homomorphism And Anti Homomorphism**

In this section, we introduce the notion of an anti image and anti pre-image of the bipolar fuzzy subgroup of a group, and discuss the properties of a bipolar fuzzy and bipolar anti fuzzy subgroup of a group under homomorphism and anti homomorphism. Throughout this section, we mean that $G_1$ and $G_2$ are finite groups ($G_1$ and $G_2$ are not necessarily commutative) $e_1, e_2$ are the identity elements of $G_1$ and $G_2$ respectively, and $xy$ we mean $x \times y$.

**Definition 3.1**

Let $f$ be a mapping from a group $G_1$ to a group $G_2$. Let $\mu$ and $\varphi$ are fuzzy subsets of $G_1$ and $G_2$ respectively, then the anti image of $\mu$ under $f$ is a fuzzy subset $\mu_\varphi$ of $G_2$ defined by for each $u \in G_2$,

$$\mu_\varphi(u) = \min \{\mu(x): x \in f^{-1}(u)\}$$

also the anti pre-image $f^{-1}(\varphi)$ of $\varphi$ under $f$ is a fuzzy subset of $G_1$ defined by for $x \in G_1$, $(f^{-1}(\varphi))^+(x) = \varphi^+(f(x)), (f^{-1}(\varphi))^-(x) = \varphi^-(f(x))$.

**Definition 3.2**

Let $f$ be a mapping from a group $G_1$ to a group $G_2$. Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets of $G_1$ and $G_2$ respectively, then the anti image of $\mu$ under $f$ is a bipolar fuzzy subset $f_\varphi(\mu) = (f(\mu)^+, f(\mu)^-)$ of $G_2$ defined by for each $u \in G_2$,

$$f_\varphi(\mu)^+(u) = \begin{cases} \min\{\mu^+(x): x \in f^{-1}(u)\} & \text{if } f^{-1}(u) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_\varphi(\mu)^-(u) = \begin{cases} \min\{\mu^-(x): x \in f^{-1}(u)\} & \text{if } f^{-1}(u) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$
also the anti pre-image \( f^{-1}(\phi) \) of \( \phi \) under \( f \) is a bipolar fuzzy subset of \( G_1 \) defined by for \( x \in G_1 \),
\[
(f^{-1}(\phi^+))(x) = \phi^+(f(x)), \quad (f^{-1}(\phi^-))(x) = \phi^-(f(x)).
\]

**Theorem 3.1**
Let \( f \) be a homomorphism from a group \( G_1 \) into a group \( G_2 \). If \( \phi = (\phi^+, \phi^-) \) is a bipolar fuzzy subset of \( G_2 \) then \( f^{-1}(\phi) = [f^{-1}(\phi^+)]^z \).

**Proof:**
Let \( \phi = (\phi^+, \phi^-) \) be a bipolar fuzzy subset of \( G_2 \) then for each \( x \in G_1 \)
i. \( [f^{-1}(\phi^+)]^x (x) = (\phi^+)^x (f(x)) = 1 - \phi^-(f(x)) = 1 - f^{-1}(\phi^-)(x) = [f^{-1}(\phi^-)]^x(x) = [f^{-1}(\phi^+)]^x \). 

ii. \( [f^{-1}(\phi^-)]^x (x) = \phi^- f(x)) = 1 - f^{-1}(\phi^-(x)) = 1 - [f^{-1}(\phi^+)]^x(x) = [f^{-1}(\phi^-)]^x = [f^{-1}(\phi^+)]^x \).

Hence, \( f^{-1}(\phi^+) = (f^{-1}(\phi^-))^c \).

**Theorem 3.2**
Let \( f \) be a homomorphism from a group \( G_1 \) into a group \( G_2 \). If \( \mu = (\mu^+, \mu^-) \) be a bipolar fuzzy subset of \( G_1 \) then i. \( f(\mu^+) = f([\mu^+]^x) \).

ii. \( f(\mu^-) = f([\mu^-]^x) \).

**Proof:**
Let \( \mu = (\mu^+, \mu^-) \) be a bipolar fuzzy subset of \( G_1 \) then for each \( x \in G_1 \), let \( f(x) = u \in G_2 \)
i. \( [f(\mu^+)]^u (x) = \max \{ (\mu^+)^x (x) : x \in f^{-1}(u) \} = \max \{ 1 - \mu^- (x) : x \in f^{-1}(u) \} = 1 - \min \{ \mu^- (x) : x \in f^{-1}(u) \} = 1 - f(\mu^-)(x) = [f(\mu^-)]^u(x) = [f(\mu^+)]^u \).

\( [f(\mu^-)]^u (x) = \max \{ (\mu^-)^x (x) : x \in f^{-1}(u) \} = \max \{ 1 - \mu^+ (x) : x \in f^{-1}(u) \} = 1 - \min \{ \mu^+ (x) : x \in f^{-1}(u) \} = 1 - f(\mu^+)(x) = [f(\mu^+)]^u(x) = [f(\mu^-)]^u \).

Hence, \( f(\mu^+) = [f(\mu^-)]^c \).

ii. \( [f(\mu^-)]^u (x) = \min \{ (\mu^-)^x (x) : x \in f^{-1}(u) \} = \min \{ 1 - \mu^+ (x) : x \in f^{-1}(u) \} = 1 - \max \{ \mu^+ (x) : x \in f^{-1}(u) \} = 1 - f(\mu^+)(x) = [f(\mu^+)]^u(x) = [f(\mu^-)]^u \).
Proof: Let \( \varphi = (\varphi^+, \varphi^-) \) be a bipolar anti fuzzy subgroup of \( G_2 \). \( \varphi^+ : G_2 \rightarrow [0,1] \) and \( \varphi^- : G_2 \rightarrow [-1,0] \) are mappings. Let \( x, y \in G_1 \).

i. \( (f^{-1}(\varphi^+))(xy) = \varphi^+(f(xy)) \)
\( \leq \max \{ \varphi^+(f(x)), \varphi^-(f(y)) \} \)
\( \leq \max \{ \varphi^+(f(x)), \varphi^-(f(y)) \} \)
\( (f^{-1}(\varphi^+))(xy) = \max \{ \varphi^+(f(x)), \varphi^-(f(y)) \} \)

ii. \( (f^{-1}(\varphi^-))(xy) = \varphi^-(f(xy)) \)
\( \leq \min \{ \varphi^+(f(x)), \varphi^-(f(y)) \} \)
\( \leq \min \{ \varphi^+(f(x)), \varphi^-(f(y)) \} \)
\( (f^{-1}(\varphi^-))(xy) = \min \{ \varphi^+(f(x)), \varphi^-(f(y)) \} \)

iii. \( (f^{-1}(\varphi^-))(x^{-1}) = \varphi^+(f(x^{-1})) \)
\( = \varphi^-(f(x^{-1})) \)
\( = \varphi^-(f(x)) \)
\( = \varphi^-((f^{-1}(\varphi^-))^{-1}(x)) \)

Hence, \( f^{-1}(\varphi^-) \) is a bipolar anti fuzzy subgroup of \( G_1 \).

REFERENCES


