Solutions To The Pell Equation $x^2 - Dy^2 = 2^k$ Where $D = r^2 s^2 + 2s$ And Recurrences

Özen Özer*, Yılmaz Çeven**

*Köyceğiz Vocational High School, Muğla University, Muğla, 48800, Turkey
Department of Mathematics, Faculty of Science and Arts, Süleyman Demirel University, Isparta, 32000, Turkey

Abstract
Let $r, s \geq 1$ and $k \geq 0$ be arbitrary integers and also $D = r^2 s^2 + 2s$ be a positive non-square integer.

In this paper, we consider the Pell equation $x^2 - Dy^2 = 2^k$ and we get all positive integer solutions of this equation for all $k \geq 0$ integers. Moreover, we derive recurrence relations on the solutions of the Pell equation $x^2 - Dy^2 = 2^k$.

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I. Introduction

The equation
\[ x^2 - Dy^2 = \mp N \] (1)
with given integers $D$, $N$ and unknowns $x$, $y$ is called Pell’s Equation. If $D$ is negative, it can have only a finite number of solutions. If $D$ is a perfect square, say $D = t^2$, the equation reduces to $(x - ty)(x + ty) = \mp N$ and there is only a finite number of solutions. The most interesting case of equation arises when $D \neq 1$ be a positive non-square integer.

For $N = 1$, the Pell equation
\[ x^2 - Dy^2 = \mp 1 \] (2)
Is known as classical Pell equation and it has infinitely many solutions $(x_n, y_n)$ for $n \in N$. There are different methods for finding the first non-trivial $(x_1, y_1)$ solution called the fundamental solution from which all other solutions are easily computed (see [2]–[3]).

Also, there are many papers in which details on Pell equations and different types of Pell’s equation are considered (see [1]–[4]–[5]–[6]).

In this paper, in the case of $D = r^2 s^2 + 2s$ where $r, s \geq 1$, we consider the Pell equation $x^2 - Dy^2 = 2^k$ when $k \geq 0$ integer and by constructing some criteria we get all positive solutions of this equation. We consider the problem in three cases:

\begin{enumerate}
  \item $(i) \ r = s = 1$
  \item $(ii) \ r \geq 2, \ s = 1$
  \item $(iii) \ r, s \geq 2$
\end{enumerate}

for $k = 0$ and $k \geq 1$ respectively. Moreover, we give numerical examples to all new constructed theorems and also by using method of \([5]\), we derive recurrences relations on the solutions of this equation.

II. Preliminary Notes

We need the following theorems for the proof of our theorems.

**Theorem 2.1.** If $N$ is a quadratic non-residue modulo $D$, then the Pell equation $x^2 - Dy^2 = N$ has no integer solution \((5)\).

**Theorem 2.2.** Let $(x_1, y_1)$ be a fundamental solution to the equation $x^2 - Dy^2 = \mp 1$. Then all positive integer solutions of the equation $x^2 - Dy^2 = \mp 1$ are given by
\[ x_n + \sqrt{D} y_n = (x_1 + \sqrt{D} y_1)^n \] (3)
with $n \geq 2$. \([3]\)

**Theorem 2.3.** Let $D$ be a positive integer, that is not a perfect square. Then the continued fraction expansion of $\sqrt{D}$ such that
\[ \sqrt{D} = [a_0; a_1, \ldots, a_l] \] where is $l(\sqrt{D}) = l$ is the period length and the $a_j$’s are given by the recursion formulas;

\[ a_0 = \left\lfloor \frac{\sqrt{D}}{2} \right\rfloor, \quad a_{l+i} = a_{l+i-2}, \quad i = 1, 2, \ldots, l \]
\[ a_0 = \sqrt{D}, \]
\[ a_t = \left[ \alpha_t \right] \quad \text{and} \quad a_{t+1} = \frac{1}{a_t - a_t} \quad \text{for } t = 0, 1, 2, \ldots \]

Recall that \( a_t = 2a_n \) and \( a_t = a_{t+1} \) for \( t \geq 1 \). The \( n^{th} \) convergent of \( \sqrt{D} \) for \( n \geq 0 \) is given by
\[
\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}.
\]

By means of the \( n^{th} \) convergent of \( \sqrt{D} \), we can give the fundamental solution of the equation \( x^2 - Dy^2 = \pm 1 \). Let \( p_{-1} = 1 \), \( p_0 = a_0 \) and \( q_{-1} = 0 \), \( q_0 = 1 \). In general
\[
p_n = a_n p_{n-1} + p_{n-2}
\]
\[
q_n = a_n q_{n-1} + q_{n-2}
\]
for \( n \geq 1 \). Then the fundamental solution of the equation \( x^2 - Dy^2 = \pm 1 \) is
\[
(x_1, y_1) = \begin{cases} (p_{l-1}, q_{l-1}) & \text{if } l \text{ is even} \\ (p_{2l-1}, q_{2l-1}) & \text{if } l \text{ is odd} \end{cases}
\]
(5)

\(([1], [2, p.154]).\)

**Theorem 2.4.** If \((u_1, v_1)\) and \((x_{n-1}, y_{n-1})\) are integer solutions of \( x^2 - Dy^2 = \pm N \) and \( x^2 - Dy^2 = \pm 1 \), respectively, then \((u_n, v_n)\) is also a positive solution of \( x^2 - Dy^2 = \pm N \), where
\[
u_n + \sqrt{D} v_n = (x_{n-1} + \sqrt{D} y_{n-1})(u_1 + \sqrt{D} v_1)
\]
(6)

for \( n \geq 2 \). ([4])

**III. The Main Results on The Pell Equation** \( x^2 - Dy^2 = 2^k \)

By using recurrence on infinite sequence of positive solutions of the Pell equation \( x^2 - Dy^2 = 2^k \) where \( D = r^2 s^2 + 2s \) with \( r, s \geq 1 \) integers and \( k \geq 0 \) is also an integer. First we consider the case \( k = 0 \), that is the classical Pell equation \( x^2 - (r^2 s^2 + 2s)y^2 = 1 \). Then, we can give following theorem.

**Theorem 3.1.** Let \( D = r^2 s^2 + 2s \) with \( r, s \geq 1 \) integers. Then the following conditions satisfy:

(a) The continued fraction expansion of \( \sqrt{D} \) is;
\[
\sqrt{D} = \begin{cases} \left[ \frac{1;1,2}{} \right] & \text{if } r = s = 1 \\
\left[ \frac{r; r, 2r}{} \right] & \text{if } r \geq 2, s = 1 \\
\left[ \frac{rs; r, 2rs}{} \right] & \text{if } r \geq 2, s \geq 2
\end{cases}
\]

(b) The fundamental solution of \( x^2 - Dy^2 = 1 \) is;
\[
(x_1, y_1) = \begin{cases} (2,1) & \text{if } r = s = 1 \\
(r^2 + 1, r) & \text{if } r \geq 2, s = 1 \\
(r^2 s + 1, r) & \text{if } r \geq 2, s \geq 2
\end{cases}
\]

(c) For \( n \geq 4 \),
\[
(x_n) = \begin{cases} \left(3(x_{n-1} + x_{n-2}) - x_{n-3}\right) & \text{if } r = s = 1 \\
\left(2r^2 + 1\right)(x_{n-1} + x_{n-2}) - x_{n-3} & \text{if } r \geq 2, s = 1 \\
\left(2r^2 s + 1\right)(x_{n-1} + x_{n-2}) - x_{n-3} & \text{if } r \geq 2, s \geq 2
\end{cases}
\]

and
\[
y_n = \begin{cases} \left(3(y_{n-1} + y_{n-2}) - y_{n-3}\right) & \text{if } r = s = 1 \\
\left(2r^2 + 1\right)(y_{n-1} + y_{n-2}) - y_{n-3} & \text{if } r \geq 2, s = 1 \\
\left(2r^2 s + 1\right)(y_{n-1} + y_{n-2}) - y_{n-3} & \text{if } r \geq 2, s \geq 2
\end{cases}
\]

**Proof:** (a) Assume that \( r = s = 1 \). By Theorem 2.3., it is easily seen that the continued fraction \( \sqrt{3} \) is
\[
\sqrt{3} = \left[ 1;1,2 \right]
\]
Now, let \( r \geq 2, s = 1 \). Then
\[
\sqrt{r^2+2} = r + \left( \sqrt{r^2+2} - r \right) = r + \frac{1}{\sqrt{r^2+2} - r} = r + \frac{1}{\sqrt{r^2+2} + r}.
\]

\[
\sqrt{r^2+2} = r + \frac{1}{\sqrt{r^2+2} - r} = r + \frac{1}{2} \frac{2}{\sqrt{r^2+2} + r}.
\]

\[
\sqrt{D} = \sqrt{r^2+2} = \left[ r, \sqrt{2} r \right].
\]

Similarly, it can be shown that
\[
\sqrt{D} = \sqrt{r^2 s + 2 s} = \left[ rs, \sqrt{2} rs \right] \text{ for } r, s \geq 2.
\]

(b) Since \((x_1, y_1) = (2, 1)\) is a fundamental solution of \(x^2 - 3y^2 = 1\), the case of \(x = 1\) is clear. Also, for \(r \geq 2, s = 1\) by using the method defined in Theorem 2.3, we get \(l = 2\), \(a_0 = r, a_1 = r\). Hence,
\[
(x_1, y_1) = (p_1, q_1) = (r^2 + 1, r) \text{ is the fundamental solution since } p_{-1} = 1, p_0 = a_0 = r, \quad q_1 = 0.
\]

\[\begin{align*}
p_1 &= a_0 p_0 + p_{-1} = r^2 + 1 + 1 = r^2 + 1, \\
q_1 &= a_0 q_0 + q_{-1} = r^0 + 0 = r,
\end{align*}\]

and (5).

Finally, we assume that \(r, s \geq 2\), by using the method defined in Theorem 2.3, we get \(l = 2\), \(a_0 = rs, a_1 = r\). Hence,
\[
(x_1, y_1) = (p_1, q_1) = (r^2 s + 1, r) \text{ is the fundamental solution since } p_{-1} = 1, p_0 = a_0 = r s.
\]

\[\begin{align*}
p_1 &= a_0 p_0 + p_{-1} = r^2 s + 1 + 1 = r^2 s + 1, \\
q_1 &= a_0 q_0 + q_{-1} = r^0 + 0 = r.
\end{align*}\]

\[
q_{-1} = 0, \quad q_0 = 1 \quad \text{and} \quad p_1 = a_0 p_0 + p_{-1} = r^2 s + 1
\]

by (4) and (5).

(c) By Theorem 2.2, we can see easily that all solutions \((x_n, y_n)\) of \(x^2 - Dy^2 = \pm 1\) can be derived from the fundamental solution \((x_1, y_1)\) of this equation. Assume that \(r = s = 1\). In a similar way in \([5]\) it can be shown by induction on \(n\) that
\[
x_n = 3(x_{n-1} + x_{n-2}) - x_{n-3},
\]
\[
y_n = 3(y_{n-1} + y_{n-2}) - y_{n-3}
\]

for \(n \geq 4\). Moreover, in a similar way, we get
\[
x_n = (2r^2 + 1)(x_{n-1} + x_{n-2}) - x_{n-3},
\]
\[
y_n = (2r^2 + 1)(y_{n-1} + y_{n-2}) - y_{n-3}
\]

where \(D = r^2 + 2\) for \(n \geq 4\) and
\[
x_n = (2rs + 1)(x_{n-1} + x_{n-2}) - x_{n-3},
\]
\[
y_n = (2rs + 1)(y_{n-1} + y_{n-2}) - y_{n-3}
\]

where \(D = r^2 s^2 + 2 s^2\) for \(n \geq 4\).

Now, we consider the general case for \(k \geq 1\). Note that we denote the integer solutions of
\[
x^2 - \left( r^2 s^2 + 2 s \right) y^2 = 2^k \text{ by } (u_n, v_n)
\]
denote the integer solutions of \(x^2 - \left( r^2 s^2 + 2 s \right) y^2 = 1\) by \((x_n, y_n)\). Then we have following theorem.

**Theorem 3.2.** Let \(r = s = 1\), that is \(D = 3\) and \(k \geq 1\) be an arbitrary integer. Define a sequence \(\{u_n, v_n\}\) of positive integers by
\[
\begin{cases}
\text{no solution} & \text{if } k \text{ is odd} \\
\left( \frac{k}{2^j+1}, \frac{k}{2^j} \right) & \text{if } k \text{ is even}
\end{cases}
\]

and, since \(k\) is even, we get
\[
\begin{align*}
u_n &= 2^j x_{n-1} + 3 \cdot 2^j y_{n-1} \\
u_n &= 2^j x_{n-1} + 3 \cdot 2^j y_{n-1}
\end{align*}
\]

where \(\{(x_n, y_n)\}\) is the sequence of positive solutions of \(x^2 - 3y^2 = 1\). Then the following conditions satisfy with \(k\) is even;
Proof: (a) Assume that $k$ is odd. Since $2$ is a quadratic non-residue mod $3$, then
\[
\left( \frac{2}{3} \right) = (-1)^k = -1.\]
By Theorem 2.1., the Pell Equation \( x^2 - 3y^2 = 2^k \) has no integer solution.

Now, let $k$ is even. Then it easily seen that
\[
\left( u_1, v_1 \right) = \left( \frac{k}{2^{n+1}}, \frac{k}{2^n} \right) \text{ is a solution of }
\]
\[
x^2 - 3y^2 = 2^k, \text{ that is }
\]
\[
u_i - Dv_i = \left( \frac{k}{2^{n+1}} \right)^2 - 3 \left( \frac{k}{2^n} \right)^2 = 2^k - 3.2^k
\]
\[
\text{Also, } (u_n, v_n) \text{ is a solution for } n \geq 2. \text{ We can prove this as follows. Recall that } (x_{n-1}, y_{n-1}) \text{ is a solution of } x^2 - 3y^2 = 1, \text{ that is, } x_{n-1} - 3y_{n-1} = 1 \tag{9}
\]
Further, we see \((u_1, v_1)\) is a solution of \(x^2 - 3y^2 = 2^k\), that is,
\[
u_i^2 - y^2 = 2^k
\]
\[
\text{using (9) and (10) , we find that }
\]
\[
u_i^2 - y^2 = \left( \frac{k}{2^{n+1}} \right)^2 x_{n-1} + \frac{3k}{2^n} \left( x_{n-1} + \frac{k}{2^n} y_{n-1} \right)^2 = 2^k x_{n-1} - 2^k \left( y_{n-1} - 3y_{n-1} \right) = 2^k
\]
Therefore, \((u_n, v_n)\) is a solution of \(x^2 - 3y^2 = 2^k\) for even $k$ integers.

(b) By Theorem 2.2. and Theorem 2.4., we get
\[
u_{n+1} = \left( u_{n+1} + \sqrt{D}v_{n+1} \right) = \left( x_{n+1} + \sqrt{D}y_{n+1} \right) \left( u_1 + \sqrt{D}v_1 \right)
\]
\[
= \left( x_1 + \sqrt{D}y_1 \right) \left( u_1 + \sqrt{D}v_1 \right)
\]
Since \((x_1, y_1) = (2,1)\) is a fundamental solution of the Pell equation \(x^2 - 3y^2 = 1\), we get
\[
u_{n+1} = 2u_n + 3v_n, \quad v_{n+1} = u_n + 2v_n
\]
for $n \geq 2$.

(c) We see as above that
\[
u_n = 2^{k-1} x_{n-1} + \frac{k}{2^n} y_{n-1}, \quad v_n = \frac{k}{2^n} x_{n-1} + \frac{k}{2^n} y_{n-1}
\]
also $u_{n+1} = 2u_n + 3v_n$. In a similar way in \([5]\), by induction on $n$ and combining these two results, it can be shown that
\[
u_n = 3(v_{n-1} + v_{n-2}) - v_{n-3}
\]
for $n \geq 4$.

Similarly, combining \((8)\) and \((8)\) results, we get
\[
u_n = 3(v_{n-1} + v_{n-2}) - v_{n-3}
\]
for $n \geq 4$.

Example: Let $r = s = 1$ and $k = 4$. Then, by Theorem 3.2., \((u_1, v_1) = (8, 4)\) is a solution of \(x^2 - 3y^2 = 2^4 = 16\), and some other solutions are; \((u_2, v_2) = (28, 16)\), \(u_3, v_3 = (104, 60)\), \((u_4, v_4) = (388, 224)\), \((u_5, v_5) = (1448, 836)\), \((u_6, v_6) = (5404, 3120)\).

Remark: Note that in Theorem 3.3. and Theorem 3.4., we will consider the case $k$ is even. When we consider the case $k$ is odd, then we find that there is a solution \((u_1, v_1)\) of \(x^2 - (r^2 + 2r) y^2 = 2^k\) and \(x^2 - (r^2 s^2 + 2s) y^2 = 2^k\) respectively, for some values of $k$, or there is no solution.

For example, \(r = 4, s = 2\) and \(k = 3\), we can not find solution of the Pell equation \(x^2 - 6y^2 = 2^5 = 8\). But for $k = 5$, we find that \((u_1, v_1) = (101, 1)\) is a solution of \(x^2 - 6y^2 = 2^5 = 32\).

Moreover, for \(r = s = 3\) and for every odd $k$, there is no solution of \(x^2 - 301y^2 = 2^k\).
Also, we can see that Keith Mathews’ “Some BC Math/ PHP Number Theory Programs”, 2013.

**Theorem 3.3.** Let \( s=1\) , \( r \geq 2\) and \( k \) be arbitrary integers with \( k \geq 1 \) is even. Define a sequence \( \{u_n, v_n\} \) of positive integers by

\[
\begin{align*}
(u_1, v_1) &= \left( \frac{r^2 + 1}{2}, \frac{r^2 + 2}{2} \right) \\
(u_{n+1}, v_{n+1}) &= \left( r^2 + (r^2 + 1), 2 \right) + \left( \frac{r^2 + 2}{2} \right) y_{n-1} - \left( \frac{r^2 + 1}{2} \right) y_{n-1}
\end{align*}
\]

and, since \( k \) is even, we get

\[
\begin{align*}
u_n &= \frac{r^2 + 1}{2} x_{n-1} + \frac{r^2 + 2}{2} r (r^2 + 2) y_{n-1} \\
v_n &= \frac{r^2 + 1}{2} x_{n-1} + \frac{r^2 + 2}{2} r (r^2 + 2) y_{n-1}
\end{align*}
\]

where \( \{(x_n, y_n)\} \) is the sequence of positive solutions of \( x^2 - (r^2 + 2) y^2 = 2^k \). Then the following conditions satisfy with \( k \) is even:

(a) \( (u_n, v_n) \) is a solution of \( x^2 - (r^2 + 2) y^2 = 2^k \) for any integer \( n \geq 1 \).

(b) For \( n \geq 2 \),

\[
u_{n+1} = (r^2 + 1) u_n + r (r^2 + 2) v_n, \quad v_{n+1} = r u_n + (r^2 + 1) v_n
\]

(c) For \( n \geq 4 \),

\[
u_n^2 - (r^2 + 2) v_n^2 = \left( \frac{r^2 + 1}{2} \right) x_{n-1} + \frac{r^2 + 2}{2} r (r^2 + 2) y_{n-1}^2 - \left( \frac{r^2 + 1}{2} \right) y_{n-1}^2
\]

\[
= x_{n-1} \left( 2^k \left( r^2 + 1 \right) - 2^k \left( r^2 + 2 \right) \right) + x_{n-1} y_{n-1} \left( 2^{k+1} r (r^2 + 1) (r^2 + 2) \right) (1 - 1)
\]

\[
= 2^k \left( x_{n-1} - 2^k \left( r^2 + 2 \right) y_{n-1}^2
\]

Therefore, \( (u_n, v_n) \) is a solution of \( x^2 - (r^2 + 2) y^2 = 2^k \) for even \( k \) integers.

Proof: (a) Assume that \( k \) is even. Then, it easily seen that \( (u_1, v_1) = \left( \frac{r^2 + 1}{2} \right) \left( \frac{r^2 + 2}{2} \right) \) is a solution of \( x^2 - (r^2 + 2) y^2 = 2^k \) since

\[
u_n^2 - D v_n^2 = \left( \frac{r^2 + 1}{2} \right)^2 - (r^2 + 2) \left( \frac{r^2 + 2}{2} \right)^2
\]

\[
= (r^2 + 1)^2 2^k - (r^2 + 2)^2 2^k = 2^k
\]

Also, \( (u_n, v_n) \) is a solution for \( n \geq 2 \). We can prove this as follows. Note that by definition, \( (x_{n-1}, y_{n-1}) \) is a solution of \( x^2 - (r^2 + 2) y^2 = 1 \), that is,

\[
x_{n-1}^2 - (r^2 + 2) y_{n-1}^2 = 1
\]

Further, we see above that \( (u_1, v_1) \) is a solution of \( x^2 - (r^2 + 2) y^2 = 2^k \), that is,

\[
u_1^2 - (r^2 + 2) v_1^2 = 2^k
\]

aplying (13) and (14), we get

Since \( (x_1, y_1) = (r^2 + 1, r) \) is a fundamental solution of the Pell equation \( x^2 - (r^2 + 2) y^2 = 1 \), we find that

\[
u_{n+1} = (r^2 + 1) u_n + r (r^2 + 2) v_n, \quad v_{n+1} = r u_n + (r^2 + 1) v_n
\]

for \( n \geq 2 \).

(c) Recall that

\[
u_n = \frac{r^2 + 1}{2} x_{n-1} + \frac{r^2 + 2}{2} r (r^2 + 2) y_{n-1}, \quad v_n = \frac{r^2 + 2}{2} x_{n-1} + \frac{r^2 + 2}{2} (r^2 + 1) y_{n-1}
\]
by (12), also \( u_{n+1} = (r^2 + 1)u_n + r(r^2 + 2)v_n \).

In a similar way in (5), by induction on \( n \) and combining these two results, it can be shown that
\[
(13) \quad u_n = (2r^2 + 1)(u_{n-1} + u_{n-2}) - u_{n-3}
\]
for \( n \geq 4 \).

Similarly, combining (12) and the two \( v_n \) results, we get
\[
(14) \quad v_n = (2r^2 + 1)(v_{n-1} + v_{n-2}) - v_{n-3}
\]
for \( n \geq 4 \).

**Example:** Let \( r = 3, \ s = 1 \) and let \( k = 2 \). Then, we get \( D = 11 \) and \( x^2 - 11y^2 = 4 \). By Theorem 3.3,
\(
(15) \quad (u_1, v_1) = (20, 6) \text{ is a solution of } x^2 - 11y^2 = 4,
\)
and some other solutions are:
\[
(16) \quad \begin{align*}
(u_2, v_2) &= (398, 120) \\
(u_3, v_3) &= (7940, 2394) \\
(u_4, v_4) &= (158402, 47760) \\
(u_5, v_5) &= (3160100, 952806)
\end{align*}
\]
\[
(17) \quad u_{n+1} = (r^2 + 1)u_n + r(r^2 + 2)v_n
\]
for \( n \geq 4 \).

\[
(18) \quad v_{n+1} = r u_n + (r^2 + 1)v_n
\]
for \( n \geq 4 \).

**Proof:** (a) Assume that \( k \) is even. Then, it easily seen that \((u_1, v_1) = \left( \frac{k}{2}, \frac{k}{2}, r \right)\) is a solution of \( x^2 - (r^2 s^2 + 2s) y^2 = 2k \) since
\[
(19) \quad u_n^2 - Dv_n^2 = \left( \frac{k}{2} \right)^2 - \left( \frac{k}{2} \right) \left( \frac{k}{2} \right) = 2^k
\]
and
\[
(20) \quad u_n^2 - (r^2 s^2 + 2s) v_n^2 = \left( \frac{k}{2} \right)^2 - \left( \frac{k}{2} \right) \left( \frac{k}{2} \right) = 2^k
\]
for \( n \geq 4 \).

Theorem 3.4. Let \( r, s \geq 2 \) and \( k \) be arbitrary integers with \( k \geq 1 \) is even. Define a sequence \( \left( u_n, v_n \right) \) of positive integers by
\[
(21) \quad \left( u_1, v_1 \right) = \left( \frac{k}{2}, \frac{k}{2}, r \right)
\]
and, since \( k \) is even, we get
\[
(22) \quad u_n = 2^k \left( r^2 s^2 + 2s \right) y_{n-1}^2 + 2^k r \left( r^2 s^2 + 2s \right) y_{n-1}^2
\]
for \( n \geq 4 \).

where \( \left( x_n, y_n \right) \) is the sequence of positive solutions of \( x^2 - (r^2 s^2 + 2s) y^2 = 1 \). Then the following conditions hold;

(a) \( (u_n, v_n) \) is a solution of
\[
(23) \quad x^2 - (r^2 s^2 + 2s) y^2 = 2^k
\]
for any integer \( n \geq 1 \).

(b) For \( n \geq 2 \),
\[
(24) \quad x_n = r u_n + (r^2 s^2 + 2s) v_n
\]
Also, \( (u_n, v_n) \) is a solution for \( n \geq 2 \). We can prove this as follows. Recall that \( \left( x_{n-1}, y_{n-1} \right) \) is a solution of \( x^2 - (r^2 s^2 + 2s) y^2 = 1 \), that is,
\[
(25) \quad x_{n-1}^2 - (r^2 s^2 + 2s) y_{n-1}^2 = 1
\]
Further, we see above that \( (u_1, v_1) \) is a solution of \( x^2 - (r^2 s^2 + 2s) y^2 = 2^k \), that is,
\[
(26) \quad u_1^2 - (r^2 s^2 + 2s) v_1^2 = 2^k
\]
applying (17) and (18), we get
\[
(27) \quad x_n^2 - (r^2 s^2 + 2s) y_n^2 = 2^k
\]
for even \( k \) integers.
(b) By Theorem 2.2. and Theorem 2.4., we get

\[ u_{n+1} + \sqrt{D}v_{n+1} = (x_1 + \sqrt{D}y_1)(u_1 + \sqrt{D}v_1) \]

\[ = (x_1 + \sqrt{D}y_1)(u_1 + v_1\sqrt{D}) \]

\[ = (x_1 + \sqrt{D}y_1)(u_n + \sqrt{D}v_n) \]

\[ u_{n+1} = (r^2s+1)u_n + r(r^2s^2+2s)v_n, \quad v_{n+1} = ru_n + (r^2s+1)v_n \]

for \( n \geq 2 \).

(c) Recall that

\[ u_n = 2^k(r^2s+1)x_{n-1} + 2^kr(r^2s^2+2s)y_{n-1}, \quad v_n = 2^krx_{n-1} + 2^kr^2(s+1)y_{n-1} \]

by (16), also

\[ u_{n+1} = (r^2s+1)u_n + r(r^2s^2+2s)v_n, \quad v_{n+1} = ru_n + (r^2s+1)v_n \]

Combining these results as (5), we find by induction on \( n \) that

\[ u_n = (2r^2s+1)(u_{n-1}+u_{n-2}) - u_{n-3}, \]

\[ v_n = (2r^2s+1)(v_{n-1}+v_{n-2}) - v_{n-3} \]

for \( n \geq 4 \).

Example: Let \( r=3 \), \( s=2 \) and let \( k=6 \). Then, we get \( D=40 \) and \( x^2 - 40y^2 = 64 \). By Theorem 3.4., \( (u_1, v_1) = (152, 24) \) is a solution of

\[ x^2 - 40y^2 = 64, \quad \text{and some other solutions are:} \]

\( (u_2, v_2) = (5768, 912) \),

\( (u_3, v_3) = (219032, 34632) \),

\( (u_4, v_4) = (8317448, 1315104) \),

\( (u_5, v_5) = (315843992, 49939320) \)

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References


