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Generalized Laplace-Finite Mellin Transform

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Abstract
Mellin transform is one kind of non-linear transformation which is widely used for its scale invariance property. It is closely connected to two sided Laplace transform. In this paper we introduce the Laplac- finite Mellin transform in the distributional generalized sense. Using Gelfand-Shilov technique the testing function space $LM_{f,u,b,c,\alpha}$ is defined. Generalized Laplace-finite Mellin transform is a Frechet space is proved. Moreover some of its topological properties are obtained.

Keywords: Laplace transforms, Mellin transform, finite Mellin transform, Laplace-finite Mellin Transform, Generalized function.

I. Introduction

Integral transforms provided a well establish and valuable method for solving initial and boundary value problem arising in several areas of physics, Applied Mathematics and Engineering. There are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin and Hankel etc. [2]. A.H. Zemanian [1] studied different integral transforms in the distributional generalized sense.

Laplace transform are widely used for solving differential and integral equation. It provides an alternative functional as description that often simplifies the process of analyzing the behavior of the system or synthesizing a view system based on a set of specification [3]. Mellin transform is important in visional & image processing. It is also used in electrical engineering for studying motor control system. Xiaohong Hu [4] developed Mellin transform technique of probability modeling for accurate solution of problems in some industrial statistics. S.M. Khairnar, R.M. Pise, et.at., [5, 6] had extended the concept of Bilateral Laplace Mellin integral transform and generalized finite Mellin transform.

Melllin transform is closely connected to Laplace transform, Fourier transform, theory of gamma function. To be successful the transform must be adopted to the form of the differential operator to be eliminated as well as to the range of interest and the associated boundary conditions. There are numerous cases for which no suitable transform exists. Motivated by the above work we aim to study the Generalization of the Laplace-finite Mellin transform in the distributional sense.

In the present paper, Laplace-finite Mellin transform is extended in the distributional generalized sense. In section 2 testing function spaces $LM_{f,u,b,c,\alpha}$ is defined. Section 3 defines distributional generalized Laplace-finite Mellin transform. $LM_{f,u,b,c,\alpha}$ is a Frachet space is proved in section 4. Topological property is proved in section 5. Section 6 concludes the conclusion.

II. Laplace-finite Mellin Transform (LM_fT)

The Laplace-finite Mellin transform (LM_fT) is defined as

$$LM_f\{f(t,x)\} = F(s,p) = \int_0^\infty \int_0^a f(t,x)\, k(t,x) dt\, dx,$$

where

$$k(t,x) = e^{-st}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)$$

2.1. Testing function space $LM_{f,u,b,c,\alpha}$

The space $LM_{f,u,b,c,\alpha}$, $\alpha \geq 0$ consists of all infinitely differentiable functions $\emptyset(x,t)$, where $0 < t < \infty, 0 < x < a,$

$$\gamma_{l,q}\, \emptyset(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| k_{u,v}(t)\lambda_{b,c}(x)\, x^{q+1} D_t^l D_x^q\, \emptyset(t,x)\right| < \infty, \quad \text{for each } l,q = 0,1,2,\dots\dots\dots$$

where
$$k_{u,v}(t) = e^{ut} \quad 0 < t < \infty$$
$$= e^{vt} \quad -\infty < t < 0$$

and
$$\lambda_{b,c}(x) = \begin{cases} x^{+b}, & 0 < x < 1 \\ x^{+c}, & 1 < x < a \end{cases}$$

2.2. Lemma

The function $e^{-st}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)$ is a member of $LM_{f,u,b,c,\alpha}$ if $-b < Rep \leq b$, for any real number c and $s>0$.

Proof: Let $\emptyset(t,x) = e^{-st}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)$

Consider

$$\gamma_{l,q}\, \emptyset(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| e^{ut}\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \emptyset(t,x) \right|$$

$$= \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| e^{ut}\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, e^{-st} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right|$$

$$= \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| e^{ut}\, \lambda_{b,c}(x)\, x^{q+1}\, (-s)^l\, e^{-st}\, [P(-p-q) x^{-p-q-1} - P(p-q)\, x^{p-q-1}] \right|,$$

where P is a polynomial in p and q.

$$= \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| e^{ut}\, (-s)^l\, e^{-st}\, [P(-p-q) x^{b-p} - P(p-q)\, x^{b+p}] \right| < \infty,$$

If as $x \to 0$,

$b-p>0$ and $b+p>0$

if $b > Rep$ $p > -b$

i.e. $Rep < b$ $-b < p$

i.e. if $-b < Rep \le b$ and for any real number c and also $s > 0$.

Thus $\emptyset(t,x) \in LM_{f,u,b,c,\propto}$ if $-b < Rep \le b$, for any real number c and $s > 0$.

III. Distributional Generalized Laplace-finite Mellin transform

For $(t,x) \in LM^*_{f,u,b,c,\propto}$, where $LM^*_{f,u,b,c,\propto}$ is the dual of $LM_{f,u,b,c,\propto}$ and $-\mathrm{b} < Rep \le b$, real number $c, s > 0$, the distributional Laplace-finite Mellin transform is defined as

$$LM_f\{f(t,x)\} = F(s,p) = \left\langle f(t,x), e^{-st} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle, \quad \cdots\cdots\cdots (3.1)$$

where for each fixed x $(0 < x < a, 0 < t < \infty)$, the right hand side of (3.1) has a sense as an application of $f \in LM^*_{f,u,b,c,\propto}$ to $e^{-st}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right) \in LM_{f,u,b,c,\propto}$.

IV. Theorem

$LM_{f,u,b,c,\propto}$; $T_{u,b,c,\alpha}$ is a Frechet space

Proof: As the family $D_{u,b,c,\alpha}$ of semi-norm $\{\gamma_{u,b,c,l}\}_{u,q,l}^{\infty} = 0$ generating $T_{u,b,c,\alpha}$ is countable, it is suffices to prove the completeness of the space $LM_{f,u,b,c,\propto}$; $T_{u,b,c,\alpha}$.

Let us consider a Cauchy sequence $\{\varphi_n\}$ in $LM_{f,u,b,c,\propto}$. Hence for a given $\in > 0$, there exist an $N = N_{u,q,l}$ such that for $m,n \ge 0$.

$$\gamma_{u,b,c,q,l}(\varphi_m - \varphi_n) = \sup_{\substack{0<x<a \\ 0<t<\infty}} \left| e^{ut}\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, (\varphi_m - \varphi_n) \right| < \infty \quad \cdots\cdots\cdots (4.1)$$

In particular for $u = q = l = 0$, for $m,n \ge n$

$$\sup_{\substack{0<x<a \\ 0<t<\infty}} \left| \lambda_{b,c}(x)\, x\, \{\varphi_m(t,x) - \varphi_n(t,x)\} \right| < \in \quad \cdots\cdots\cdots (4.2)$$

consequently, for fixed (t,x) over $0 < x < a, 0 < t < \infty, \{\varphi_m(t,x)\}$ is a numerical Cauchy sequence. Let $\varphi(t,x)$ be the point wise limit of $\{\varphi_m(t,x)\}$.

Using (4.2), we can easily deduce that $\{\varphi_m\}$ converges to φ uniformly on $0 < x < a, 0 < t < \infty$. Thus φ is continuous.

Moreover, repeated use of (4.1) for different value of u, q, l yields that φ is smooth i.e. $\varphi \in E$, where E is the class of infinitely differentiable function defined over $0 < x < a, 0 < t < \infty$.

Further from (4.1) we get,

$$\gamma_{u,b,c,q,l}(\varphi_m) \le \gamma_{u,b,c,q,l}(\varphi_n) + \varepsilon \quad \forall \quad m \ge N$$
$$\le C_{l,q}\, A^u\, u^{u\alpha} + \varepsilon$$

Taking m → ∞ and ε is arbitrary we get,

$$\gamma_{u,b,c,q,l}(\varphi) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| e^{ut}\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \varphi(t,x) \right| \le C_{l,q}\, A^u\, u^{u\alpha}$$

Hence $\varphi \in LM_{f,u,b,c,\propto}$ and it is the $T_{u,b,c,\alpha}$ limit of φ_m by (4.1) again. This proves the completeness of $LM_{f,u,b,c,\propto}$.

Therefore $LM_{f,u,b,c,\propto}$; $T_{u,b,c,\alpha}$ is a Frechet space.

V. Theorem

The space $D(I)$ is a subspace of $LM_{f,u,b,c,\propto}$.

Proof: For $\varphi(t,x) \in D(I)$, set

$$L = \sup_{\substack{0<x<a \\ 0<t<\infty}} \{t : (t,x) \in supp\, \varphi\} \quad and$$

$$C_{lq} = \sup_{I} \left| e^{ut}\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \varphi(t,x) \right|$$

$$\gamma_{u,b,c,q,l}\, \varphi(t,x) = \sup_{I} \left| e^{ut}\, \lambda_{b,c}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \varphi(t,x) \right| \le C_{lq} L^u$$

$$= C_{lq} \left(\frac{L}{Au^{\alpha}} \right)^u A^u u^{u\alpha} \quad \cdots\cdots\cdots (5.1)$$

Since $\left(\frac{L}{Au^{\alpha}}\right)^u \le 1$ iff $k \ge \left(\frac{L}{A}\right)^{\frac{1}{\alpha}}$,

Define $u_0 = \left[\left(\frac{L}{A}\right)^{\frac{1}{\alpha}} \right] + 1$,

where $[t]$ denotes the Gaussian symbol, that is the greatest integer not exceeding t.

Therefore for $u > u_0$, we have

$$\gamma_{u,b,c,q,l}\, \varphi(t,x) \le C_{lq} A^u u^{u\alpha} \quad \cdots\cdots\cdots (5.2)$$

If $u \le u_0$, let us write,

$$c = max\left\{\frac{L}{A}, \left(\frac{L}{A2^{\alpha}}\right)^2, \dots\dots, \left(\frac{L}{Au_0^{\alpha}}\right)^{u_0}\right\}$$

Then again from (5.1)

$$\gamma_{u,b,c,q,l}\ \varphi(t,x) \le CC_{lq}A^u u^{u\alpha} \qquad \cdots\cdots\cdots (5.3)$$

Hence the inequalities (5.2) and (5.3) together yield.

$$\gamma_{u,b,c,q,l}\ \varphi(t,x) \le C'_{lq}A^u u^{u\alpha} \qquad \forall\ u \ge 0$$

where $C'_{lq} = CC_{lq}$

Implying that $\varphi \in LM_{f,u,b,c,\propto}$.

Consequently, $D(I) \subset LM_{f,u,b,c,\propto}$.

VI. Conclusion

Laplace-finite Mellin transform is extended in the distributional generalized sense. Testing function space using Gelfand Shilov technique and distributional generalized Laplace-finite Mellin transform is defined. Laplace-finite Mellin transform is a Frechet space and its topological property is proved.

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