Effect of first order chemical reaction in a vertical double-passage channel

J. Prathap Kumar, J.C. Umavathi, and Jagtap Sharadkumar
Department of Mathematics, Gulbarga University, Gulbarga, Karnataka – 585 106, India

ABSTRACT:
Fully developed laminar mixed convection flow in a vertical channel in the presence of first order chemical reactions has been investigated. The channel is divided into two passages by means of a thin, perfectly plane conducting baffle and hence the velocity, temperature and concentration will be individual in each stream. The coupled, nonlinear ordinary differential equations are solved analytically using regular perturbation method valid for small values of Brinkman number. The effects of thermal Grashof number, mass Grashof number, Brinkman number and chemical reaction parameter on the velocity, temperature and concentration fields at different positions of the baffle are presented and discussed in detail. The increase in thermal Grashof number, mass Grashof number and Brinkman number enhances the flow, whereas the chemical reaction parameter suppress the flow at all baffle positions.

I. INTRODUCTION:
Convective heat transfer in a vertical channel has practical importance in many engineering systems. Examples are solar energy collection, the design of heat exchangers and the cooling of electronic systems. A convection situation, in which the effects of both forced and free convection are significant is commonly referred to as mixed convection or combined convection. The effect is especially pronounced in situations where the forced fluid flow velocity is moderate and/or the temperature difference is very large. In mixed convection flows, the forced convection effects and the free convection effects are of comparable magnitude. Thus, in this situation, both the forced and free convection occurs simultaneously, that is mixed convection occurs in which the effect of buoyancy forces on a forced flow or the effect of forced flow on buoyant flow is significant. Merkin [1, 2] has studied dual solutions occurring in the problem of the mixed convection flow over a vertical surface through a porous medium with constant temperature for the case of opposing flow. Aly et al. [3] and Nazar and Pop [4] have investigated the existence of dual solutions for mixed convection in porous medium.

The techniques of mixed convection to enhance heat transfer is an important main objective in numerous engineering applications of the micro and nano fluids including pumping systems, chemical catalytic reactors, electronic system cooling, plate-fin heat exchangers, etc. Moreover, the arrangement of baffles may be used to cool down the temperature in the passage for the thermally developed flow. When the channel is divided into several passages by means of plane baffles, as usually occurs in heat exchangers or electronic equipment, it is quite possible to enhance the heat transfer performance between the walls and fluid by the adjustment of each baffle position and strengths of the separate flow streams. In such configurations, perfectly conductive and thin baffles may be used to avoid significant increase of the transverse thermal resistance. Stronger streams may be arranged to occur within the passages near the channel wall surfaces in order to cool or heat the walls more effectively. Even though the subject of channel flow have been investigated extensively, few studies have so far evaluated for these effects. Recently, Prathap Kumar et al. [5, 6] and Umavathi [7] analyzed the free convection in a vertical double passage wavy channel for both Newtonian and non-Newtonian fluids.

The growing need for chemical reactions in chemical and hydro metallurgical industries requires the study of heat and mass transfer in the presence of chemical reactions. There are many transport processes that are governed by the simultaneous action of buoyancy forces due to both thermal and mass diffusion in the presence of chemical reaction effect. These processes are observed in nuclear reactor safety and combustion systems, solar collectors as well as chemical and metallurgical engineering. Das et al. [8] have studied the effects of mass transfer on the flow started impulsively past an infinite vertical plate in the presence of wall heat flux and chemical reaction. Muthucumaraswamy and Ganeshan [9, 10] have studied the impulsive motion of a vertical plate with heat flux/mass and diffusion of chemically reactive species. Seddeek [11] has studied the finite element method for the effects of chemical reaction, variable viscosity, thermophoresis, and heat generation/absorption on a
boundary layer hydro magnetic flow with heat and mass transfer over a heat surface.

When the previous studies are reviewed, one can see there was no much study performed on heat and mass transfer on a vertical enclosure with baffle. Therefore, an analysis is made to understand heat and mass transfer in a vertical channel by introducing a perfectly thin baffle for viscous fluid in the presence of first order chemical reaction.

II. MATHEMATICAL FORMULATION:

Consider a steady, two-dimensional laminar fully developed mixed convection flow in an open ended vertical channel filled with pure viscous fluid and is concentrated. The X-axis is taken vertically upward, and parallel to the direction of buoyancy, and the Y-axis is normal to it (see Fig. 1). The walls are maintained at a constant temperature. The fluid properties are assumed to be constant. The channel is divided into two passages by means of thin, perfectly conducting plane baffle and each stream will have its own pressure gradient and hence the velocity will be individual in each stream.

\[
\begin{align*}
\rho g \beta \left( T_i - T_{in} \right) + \rho g \beta_c (C_i - C_{in}) - \frac{\partial P}{\partial X} + \mu \frac{d^2 U_i}{dY^2} &= 0 \\
K \frac{d^2 T_i}{dY^2} + \mu \left( \frac{dU_i}{dY} \right)^2 &= 0 \\
D \frac{d^2 C_i}{dY^2} - K C_i &= 0
\end{align*}
\]

subject to the boundary conditions on velocity, temperature and concentration as

\[
\begin{align*}
U_1 &= 0, \quad T_i = T_{in}, \quad C_i = C_{in} \quad \text{at} \quad Y = -h \\
U_2 &= 0, \quad T_i = T_{in}, \quad C_i = C_{in} \quad \text{at} \quad Y = h \\
U_1 &= 0, \quad U_2 = 0, \quad T_i = T_{in}, \quad \frac{dT_i}{dY} = \frac{dT_2}{dY} \quad \text{at} \quad C_i = C_{in}, \quad C_2 = C_{in} \quad \text{at} \quad Y = h^*
\end{align*}
\]

Introducing the following non-dimensional variables

\[
\begin{align*}
u_i &= \frac{U_i}{U_0}, \quad \theta_i = \frac{T_i - T_{in}}{T_{in} - T_{w}}, \quad Gr = \frac{g \beta_c \Delta T n^3}{\nu^2}, \\
Gr_c &= \frac{g \beta_c \Delta C_c n^3}{\nu^2}, \quad Gr_C = \frac{g \beta_c \Delta C_c n^3}{\nu^2}, \quad Re = \frac{\bar{U}_i h}{\nu}, \\
Br &= \frac{\bar{U}_i^2 \mu}{K \Delta T}, \quad p = \frac{h^2}{\mu} \frac{\partial P}{\partial X}, \quad i = 1, 2, \quad \Delta T = T_{w_1} - T_{w_2}, \\
\phi_i &= \frac{C_i - C_{oi}}{C_i^* - C_{oi}}, \quad \phi_C = \frac{C_C - C_{oi}}{C_C - C_{oi}}, \quad \Delta C_i = C_i^* - C_{oi}, \\
\Delta C_2 &= C_2^* - C_{oi}, \quad Y = \frac{y}{h}, \quad y^* = \frac{y^*}{h}, \quad n_i = \frac{C_i - C_{oi}}{C_i^* - C_{oi}}, \\
n_2 &= \frac{C_2 - C_{oi}}{C_2^* - C_{oi}}.
\end{align*}
\]

One obtains the momentum, energy and concentration equations in the non-dimensional form in stream-I and stream-II as

Stream-I

\[
\begin{align*}
\frac{d^2 u_i}{dy^2} + Gr \theta_i + Gr_C \phi_i - p &= 0 \\
\frac{d^2 \theta_i}{dy^2} + Br \left( \frac{du_i}{dy} \right)^2 &= 0 \\
\frac{d^2 \phi_i}{dy^2} - \alpha_i^2 \phi_i &= 0
\end{align*}
\]

Stream-II

\[
\begin{align*}
\frac{d^2 u_2}{dy^2} + Gr_2 \theta_2 + Gr_C \phi_2 - p &= 0 \\
\frac{d^2 \theta_2}{dy^2} + Br \left( \frac{du_2}{dy} \right)^2 &= 0 \\
\frac{d^2 \phi_2}{dy^2} - \alpha_2^2 \phi_2 &= 0
\end{align*}
\]
where \( \alpha_1 = h \sqrt{K_1 \over D} \), \( \alpha_2 = h \sqrt{K_2 \over D} \), \( GR = {Gr \over Re} \) and \( GR_{C1} = {GC_1 \over Re} \), \( GR_{C2} = {GC_2 \over Re} \).

II. SOLUTIONS:

Equations (9), (10), (12) and (13) are first order equations as given in equations (26) and (27) are given as follows

Stream-I

Zeroth order equations

\[
\frac{d^2 \theta_{10}}{dy^2} = 0
\]

(18)

First order equations

\[
\frac{d^2 \theta_{11}}{dy^2} + \left( \frac{du_{10}}{dy} \right)^2 = 0
\]

(20)

\[
\frac{d^2 u_{11}}{dy^2} + GR_{C1} \theta_{11} = 0
\]

(21)

Stream-II

Zeroth order equations

\[
\frac{d^2 \theta_{20}}{dy^2} = 0
\]

(22)

First order equations

\[
\frac{d^2 \theta_{21}}{dy^2} + \left( \frac{du_{20}}{dy} \right)^2 = 0
\]

(24)

\[
\frac{d^2 u_{21}}{dy^2} + GR_{C2} \theta_{21} = 0
\]

(25)

The corresponding zeroth order and first order boundary conditions are

\[
\begin{align*}
\theta_{10} &= 0, \quad \theta_{20} = 1 \text{ at } y = -1 \\
\theta_{10} &= 0, \quad \theta_{20} = 0 \text{ at } y = 1 \\
\theta_{10} &= 0, \quad \theta_{20} = 0 \text{ at } y = 1
\end{align*}
\]

(26)

\[
\begin{align*}
\frac{d \theta_{10}}{dy} &= 0, \quad \frac{d \theta_{20}}{dy} = 0 \text{ at } y = -1 \\
\frac{d \theta_{10}}{dy} &= 0, \quad \frac{d \theta_{20}}{dy} = 0 \text{ at } y = 1
\end{align*}
\]

(27)

The solutions of zeroth and first order equations (18) to (25) using the boundary conditions as given in equations (26) and (27) are given as follows

Stream-I

\[
\begin{align*}
\psi &= c_1 y + c_2 \\
u_{10} &= A_1 + A_2 y + r_1 y^2 + r_2 y^3 + r_3 Cosh(\alpha_1 y) \\
&\quad + r_4 Sinh(\alpha_1 y) \\
u_{20} &= A_1 + A_2 y + r_3 y^2 + r_4 y^3 + r_5 Cosh(\alpha_2 y) \\
&\quad + r_6 Sinh(\alpha_2 y)
\end{align*}
\]

(28)

(29)

(30)

(31)

First order equations

Stream-I

\[
\begin{align*}
\psi &= c_1 y + c_2 \\
u_{11} &= E_1 y + E_2 y^2 + s_1 y^3 + s_2 y^4 + s_3 y^5 + s_4 y^6 \\
&\quad + s_5 Cosh(2\alpha_1 y) + s_6 Sinh(2\alpha_1 y) + s_7 Cosh(\alpha_1 y) \\
&\quad + s_8 Sinh(\alpha_1 y) + s_{10} y Cosh(\alpha_1 y) + s_{11} y Sinh(\alpha_1 y) \\
&\quad + s_{12} y^2 Cosh(\alpha_1 y) + s_{13} y^2 Sinh(\alpha_1 y)
\end{align*}
\]

(32)

(33)

Stream-II

\[
\begin{align*}
\psi &= c_1 y + c_2 \\
u_{21} &= E_1 y + E_2 y + w_1 y^2 + w_2 y^3 + w_3 y^4 + w_4 y^5 + w_5 y^6 + w_6 y^7 + w_7 Sinh(\alpha_2 y) + w_{10} Cosh(2\alpha_2 y) + w_{11} y Cosh(\alpha_2 y) + w_{12} y Sinh(\alpha_2 y) \\
&\quad + w_{13} y^2 Cosh(\alpha_2 y) + w_{14} y^2 Sinh(\alpha_2 y)
\end{align*}
\]

(34)

(35)

Solutions of equations (11) and (14) can be obtained directly and are given as follows

\[
\phi = B_1 Cosh(\alpha_1 y) + B_2 Sinh(\alpha_1 y)
\]

(36)

\[
\phi = B_1 Cosh(\alpha_2 y) + B_2 Sinh(\alpha_2 y)
\]

(37)
III. RESULTS AND DISCUSSIONS:

The velocity, temperature, and concentration fields for a viscous fluid in a vertical channel in the presence of first order chemical reaction containing a thin conducting baffle is studied analytically. The non-linear coupled ordinary differential equations governing the flow have been solved by regular perturbation method. The thermal Grashoff number \( GR_T \), mass Grashoff numbers \( GR_p (= GR_{bc} = GR_{ac}) \), pressure gradient \( p \), chemical reaction parameter \( \alpha \) are fixed as 5, 5, -5, and 1 respectively, for all the graphs except the varying one.

The effect of thermal Grashoff number \( GR_T \) on the velocity and temperature is shown in Figures 2a,b,c and 3a,b,c respectively at three different baffle positions. As the thermal Grashoff number increases, the velocity increases near the left (hot) wall, and also at the right (cold) wall at all the baffle positions in both the streams. The maximum point of velocity is in stream-II for the baffle position at \( y = -0.8 \), where as it is in stream-I for the baffle position at \( y = 0.0 \) and 0.8. From Figure 3a,b,c it is seen that when the baffle position is near the hot wall, the temperature increases in stream-I where as in stream-II it increases up to \( y = 0.5 \) and decreases from \( y = 0.5 \) as seen in Figure 3a. The temperature increases continuously in both the streams when the baffle is at the center of the channel and at the right wall as seen in Figure 3b and 3c respectively. The increase in temperature with increase in thermal Grashoff number is obvious. As the increase in thermal Grashof number increases the buoyancy force and hence increases the velocity and temperature fields.

The effect of mass Grashoff number \( GR_p (= GR_{bc} = GR_{ac}) \) on the flow is shown in Figure 4a,b,c and Figure 5a,b,c at all the baffle positions. The effect of \( GR_p \) is to increases the velocity and temperature in both the streams at all baffle positions. When the baffle position is near the left and right walls, there is no much effect of mass Grashoff number on the flow field. There is a distinct increase in velocity and temperature field in stream-II when the baffle position is near the left wall, in stream-I and in stream-II when the baffle position is in the middle of the channel and in stream-I when the baffle position is near the right wall. From Figure 5a,b,c it is seen that there is reversal of temperature field in stream-II when the baffle position is near the left and middle of channel, where as there is no such reversal of the temperature field when the baffle position is at the right wall. This is due to the fact that the left wall is at higher temperature when compared to right wall. Further increase in mass Grashoff number implies increase in concentration buoyancy force and hence increases the flow field. The similar result was also observed by Muturaj and Srinivas [12] without inserting the baffle in a vertical wavy channel.

The effect of Brinkman number \( Br \) on the flow field is shown in Figures 6a,b,c and 7a,b,c. As the Brinkman number increases both the velocity and temperature increases in both the streams at all the baffle positions. However the increase in velocity and temperature is highly significant in stream-I when the baffle position is near the left wall, in both the streams when the baffle position is in the middle of the channel and in stream-I when the baffle position is at the right wall. This is due to the fact that as the Brinkman number increases the viscous dissipation also increases which results in the increase of the flow field.

The effect of first order chemical reaction parameter \( \alpha (= \alpha_b = \alpha_c) \) on the velocity, temperature and concentration is shown in Figures 8a,b,c, 10a,b,c, and 11a,b,c respectively. As \( \alpha \) increases the velocity, temperature and concentration decreases in both the streams at all the baffle positions. This is due to the facts that increase in \( \alpha \) increase the concentration of the fluid which results in the reduction of heat and mass transfer. The increase of chemical reaction parameter was to reduce the velocity, concentration field as observed by Shivarir and Anand Rao [13] for unsteady MHD free convetion flow past a vertical porous plate.

IV. CONCLUSIONS:

The effect of first order chemical reaction on heat and mass transfer of viscous fluid in a vertical double-passage channel by inserting perfectly thin conducting baffle was analysed. According to the results following conclusions can be drawn.

[1] Increasing the values of thermal Grashoff number increases the velocity and temperature in both the streams at all the baffle positions.

[2] Increasing the value of mass Grashoff number increases the velocity in both the streams at all the baffle positions. The temperature increases in stream-I where as it reverses its direction near the right wall when the baffle position is at the left wall. The temperature field increase in both the streams as mass Grashoff number increases, when the baffle position is in the middle of the channel and at the right wall.

[3] With the increase in Brinkman number, increases the flow field in both the streams at all the baffle positions.

[4] The increase in the chemical reaction parameter decreases the velocity, temperature and concentration in both the streams at all the baffle positions.

[5] By inserting the baffle the heat transfer can be enhanced depending on the position of the baffle.
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Figure 2. velocity distribution for different values of thermal Grashoff number $G_{RT}$ at (a) $y^*=0.8$ (b) $y^*=0.0$ (c) $y^*=0.8$
Figure 3. Temperature profile for different values of Thermal Grashoff number $GR_T$ at (a) $y^*=0.8$ (b) $y^*=0$ (c) $y^*=0.8$

Figure 4. Velocity profile for different values of mass Grashoff number $GR_c$ at (a) $y^*=-0.8$ (b) $y^*=0$ (c) $y^*=0.8$
Figure 5. Temperature profile for different values of mass Grashoff number $GR_c$ at (a) $y^*= -0.8$ (b) $y^*= 0$ (c) $y^*= 0.8$

Figure 6. Velocity for different values of Brinkman number $Br$
(a) $y^*= -0.8$ (b) $y^*= 0$ (c) $y^*= 0.8$
Figure 7. Temperature profile for different values of Brinkman number Br at (a) $y^*= -0.8$ (b) $y^* = 0$ (c) $y^* = 0.8$

Figure 8. Velocity profile for different values of chemical reaction parameter $\alpha$ at (a) $y^* = -0.8$ (b) $y^* = 0$ (c) $y^* = 0.8$
Figure 9. Temperature profile for different values of chemical reaction parameter $\alpha$ at (a) $y^*=0.8$ (b) $y^*=0$ (c) $y^*=0.8$

Figure 10. Concentration profile for different values of chemical reaction parameter $\alpha$ at (a) $y^*=0.8$ (b) $y^*=0$ (c) $y^*=0.8$
NOMENCLATURE:

- $Br$: Brinkman number
- $C_p$: specific heat at constant pressure
- $C_1, C_2$: the concentration in stream-I and stream-II
- $C_{ref}, C_{ref}$: reference concentrations
- $D$: diffusion coefficients
- $g$: acceleration due to gravity
- $Gr$: Grashoff number $\left( \frac{h^3 g \beta \Delta T}{\nu^2} \right)$
- $Gr_{c1}, Gr_{c2}$: modified Grashoff number
- $h$: channel width
- $h^*$: width of passage
- $k$: thermal conductivity of fluid
- $p$: non-dimensional pressure gradient
- $Re$: Reynolds number $\left( \frac{U h}{\nu} \right)$
- $U_1$: reference velocity
- $U_1, U_2$: dimensional velocity distributions
- $u_i, u_2$: nondimensional velocities in stream-I, stream-II
- $T_1, T_2$: dimensional temperature distributions
- $T_{1i}, T_{2i}$: temperatures of the boundaries
- $y^*$: baffle position

GREEK SYMBOLS

- $\alpha_1, \alpha_2$: chemical reaction parameters
- $\beta_r$: coefficients of thermal expansion
- $\beta_c$: coefficients of concentration expansion
- $\Delta T, \Delta C$: difference in temperatures and concentration
- $\theta$: non-dimensional temperature
- $\gamma$: kinematics viscosity
- $\phi_1, \phi_2$: non-dimensional concentrations
- $\rho$: density
- $\mu$: viscosity

SUBSCRIPTS

- $i$: refer quantities for the fluids in stream-I and stream-II, respectively.

APPENDIX:

- $B_2 = \frac{\text{Sh}(\alpha, y^*) + n \text{Sinh}(\alpha)}{\text{Sh}(\alpha)} - 
\frac{\text{Cosh}(\alpha) \text{Cosh}(\alpha) \text{Sinh}(\alpha)}{\text{Cosh}(\alpha)}$
- $B_1 = \frac{n \text{Cosh}(\alpha) \text{Cosh}(\alpha) \text{Sinh}(\alpha)}{\text{Cosh}(\alpha) \text{Cosh}(\alpha) \text{Sinh}(\alpha)}$
- $B_2 = \frac{\text{Sh}(\alpha)}{\text{Sh}(\alpha) - n \text{Sinh}(\alpha)}$
- $B_1 = \frac{1}{\text{Sh}(\alpha) - \text{Sinh}(\alpha)}$
- $c_4 = \frac{1}{2}, r_1 = \left( \frac{p - Gr_{c1}}{2} \right), r_2 = \frac{Gr_{c1}}{6}$
- $c_4 = \frac{1}{2}, r_1 = \frac{Gr_{c1}}{2}, r_2 = \frac{Gr_{c1}}{6}$
- $r_3 = \frac{Gr_{c2}}{2}, r_4 = \frac{Gr_{c2}}{6}$
- $r_1 = \frac{Gr_{c2} B_{c1}}{2}, r_2 = \frac{Gr_{c2} B_{c1}}{6}$
- $A_3 = A_1 - r_1 - r_2 - r_3 \text{Cosh}(\alpha) + r_1 \text{Sinh}(\alpha)$
- $A_4 = -A_3 - r_1 - r_2 - r_3 \text{Cosh}(\alpha) - r_1 \text{Sinh}(\alpha)$
- $A_5 = \left( r_1 (y^* - 1) + r_2 (y^* + 1) + \frac{r_3}{2} \text{Cosh}(\alpha) - \text{Cosh}(\alpha) \right) + r_4 \left( \text{Sinh}(\alpha) \right)
- A_i = \left( \frac{r_1 (y^* - 1) + r_2 (y^* + 1) + r_3 (\text{Cosh}(\alpha) - \text{Cosh}(\alpha)) + r_4 (\text{Sinh}(\alpha) + \text{Sinh}(\alpha))}{2} \right)
- p_1 = -\frac{2 A_1 r_3}{3}, p_2 = -\frac{2 A_1 r_3}{3}$
\[ s_9 = \frac{2A_0 r \alpha^2 - 8 \sigma R r \alpha_2 + 36 \sigma n \rho}{\alpha_2^2}, \quad s_{10} = \frac{-4r_0 \alpha \alpha - 24 r \sigma}{\alpha_2^2}, \quad s_{11} = \frac{-4r_0 \alpha \alpha - 24 r \sigma}{\alpha_2^2}, \quad s_{12} = \frac{6 \alpha \rho}{\alpha_2}, \quad s_{13} = \frac{6 \alpha \rho}{\alpha_2}, \]

\[ G_9 = -(p_1 - p_2 + p_3 - p_4 + p_5 + p_6 \cosh(2 \alpha) - p_7 \sinh(2 \alpha) + p_8 \cosh(\alpha)) \]

\[ -p_9 \sinh(\alpha) - p_10 \cosh(\alpha) + p_11 \sinh(\alpha) + p_12 \cosh(\alpha) - p_13 \sinh(\alpha) \]

\[ G_{10} = -s_1 \sinh(\alpha) + s_2 \cosh(2 \alpha) + s_3 \sinh(\alpha) + s_4 \cosh(\alpha) \]

\[ G_{11} = G_1 + G_2 \alpha + s_5 \cosh(2 \alpha) + s_6 \sinh(2 \alpha) + s_7 \cosh(\alpha) + s_8 \sinh(\alpha) \]

\[ -(p_2 \cosh(2 \alpha) - p_3 \sinh(2 \alpha) + p_4 \cosh(\alpha)) - p_5 \cosh(\alpha) + p_6 \sinh(\alpha) - p_7 \cosh(\alpha) - p_8 \sinh(\alpha) \]

\[ G_{12} = 2s_9 \alpha^2 + 3s_4 \cosh(2 \alpha) + 5s_6 \cosh(2 \alpha) - 2s_3 \sinh(2 \alpha) + s_2 \cosh(\alpha) + s_4 \sinh(\alpha) + s_5 \cosh(\alpha) + s_6 \sinh(\alpha) \]

\[ \alpha \cosh(\alpha) + \alpha \sinh(\alpha) \]

\[ G_{13} = -R_1 - R_2 + R_3 - R_4 + R_5 - R_6 + R_7 + R_8 \cosh(2 \alpha) - R_9 \sinh(2 \alpha) + R_{10} \cosh(\alpha) \]

\[ -R_1 \sinh(\alpha) - R_2 \cosh(\alpha) + R_3 \sinh(\alpha) + R_4 \cosh(\alpha) - R_5 \sinh(\alpha) \]

\[ G_{14} = (w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 \cosh(2 \alpha) + w_9 \sinh(2 \alpha) + w_{10} \cosh(\alpha) \]

\[ +w_11 \sinh(\alpha) + w_{12} \cosh(\alpha) + R_11 \sinh(\alpha) + w_{13} \cosh(\alpha) \]

\[ G_{15} = w_1 \cosh(\alpha) + w_2 \cosh(\alpha) + w_3 \cosh(\alpha) + w_4 \cosh(\alpha) + w_5 \cosh(\alpha) + w_6 \cosh(\alpha) + w_7 \cosh(\alpha) + w_8 \cosh(\alpha) + w_9 \cosh(\alpha) + w_{10} \cosh(\alpha) \]

\[ +w_11 \cosh(\alpha) + w_{12} \sinh(\alpha) + w_{13} \cosh(\alpha) + w_{14} \cosh(\alpha) + w_{15} \cosh(\alpha) \]

\[ E_5 = (G_5 - G_{13})/(1 + y^*), \quad E_7 = (G_{14} - G_{16})/(1 - y^*), \quad E_9 = G_{13} + E_3, \quad E_8 = E_{14} - E_2 \]