Some Double Finite Integrals Involving the Hypergeometric Functions and $\overline{H}$-Function With Applications

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Abstract

The aim of this paper is to evaluate four finite double integrals involving the product of two hypergeometric functions and $\overline{H}$-function. At the end we give an application of our findings by connecting them with the Riemann-Liouville type of fractional integral operator. The results obtained by us are basic in nature and are likely to find useful applications in several fields notably electric network, probability theory and Statistical mechanics.

Key words: Double finite integral, Riemann-Liouville fractional integral operator, $\overline{H}$-function.
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I. Introduction

The $\overline{H}$-function occurring in the paper will be defined and represented by Buschman and Srivastava [2] as follows:

$$\overline{H}_{P,Q}^{M,N}[z]=\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \quad (1.1)$$

where $$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \quad (1.2)$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,...,p)$ and $b_j(j=1,...,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,...,P),\beta_j\geq 0(j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,...,N)$ and $B_j(j=N+1,...,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$-function given by equation (1.1) have been given by (Buschman and Srivastava[2]).

$$\Omega\equiv\sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_jB_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0 \quad (1.3)$$

and $$\left|\arg(z)\right|<\frac{1}{2}\pi\Omega \quad (1.4)$$

The behavior of the $\overline{H}$-function for small values of $|z|$ follows easily from a result recently given by (Rathie [4],p.306,eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}[z]=0\left(|z|^{\gamma}\right),\gamma=\min_{1\leq j\leq N}\left[\operatorname{Re}\left(b_j/\beta_j\right)\right],|z|\to 0 \quad (1.5)$$

If we take $A_j=1(j=1,...,N)$, $B_j=1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [3].

We shall use the following notation:

$$A^*=\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P} \text{ and } B^*=\left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q}$$

II. Preliminary Results

The following known results [6] will be required in the proof:

$$\int_0^1 x^{\rho-1}(1-x)^{\rho}\left[1+ax+b(1-x)\right]^{-2\rho-1} {}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]dx$$

$$=2^{\alpha+\beta-2\rho}\frac{\Gamma\left(\rho-\frac{\alpha}{2}-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\Gamma(\rho)}{(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)}$$

$$\cdot\left[\frac{(2\rho-\alpha+\beta)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\rho-\frac{\alpha}{2}-1\right)\Gamma\left(\rho-\frac{\beta}{2}+\frac{1}{2}\right)}-\frac{(2\rho+\alpha+\beta)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\rho-\frac{\beta}{2}+1\right)\Gamma\left(\rho-\frac{\alpha}{2}+\frac{1}{2}\right)}\right] \quad (2.1)$$

Where $\mathrm{Re}(\rho)>0, \mathrm{Re}(2\rho-\alpha-\beta)>0, a$ and b are constants, such that the expression $[1+ax+b(1-x)]$ is not zero and $0\le x\le 1$.

$$\int_0^1 x^{\rho-1}(1-x)^{\rho}\left[1+ax+b(1-x)\right]^{-2\rho+1} {}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta)}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]dx$$

$$=2^{\alpha+\beta-2\rho-1}\frac{\Gamma\left(\rho-\frac{\alpha}{2}-\frac{\beta}{2}-1\right)\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma(\rho-1)}{(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)}$$

$$\cdot\left[\frac{(2\rho-\alpha+\beta-2)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\rho-\frac{\alpha}{2}\right)\Gamma\left(\rho-\frac{\beta}{2}-\frac{1}{2}\right)}+\frac{(2\rho+\alpha-\beta)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\rho-\frac{\beta}{2}\right)\Gamma\left(\rho-\frac{\alpha}{2}-\frac{1}{2}\right)}\right] \quad (2.2)$$

Where $\mathrm{Re}(\rho)>0, \mathrm{Re}(2\rho-\alpha-\beta)>0, a$ and b are constants, such that the expression $[1+ax+b(1-x)]$ is not zero and $0\le x\le 1$.

$$\int_0^{\pi/2} e^{i(2w+1)\pi\theta}(\sin\theta)^{w-1}(\cos\theta)^{w-1} {}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}\cos\theta\right]d\theta$$

$$=\frac{e^{i\pi(w+1)/2}\Gamma(w)\Gamma\left(w-\frac{\alpha'-\beta'}{2}\right)\Gamma\left(\frac{\alpha'-\beta'}{2}+1\right)}{2^{2w-\alpha'-\beta'+2}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}$$

$$\cdot\left[\frac{(2w-\alpha'-\beta')\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)}{\Gamma\left(w-\frac{\alpha'}{2}+1\right)\Gamma\left(w-\frac{\beta'-1}{2}\right)}-\frac{(2w+\alpha'-\beta')\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)}{\Gamma\left(w-\frac{\beta'}{2}+1\right)\Gamma\left(w-\frac{\alpha'-1}{2}\right)}\right] \quad (2.3)$$

Where $\mathrm{Re}(w)>0, \mathrm{Re}(2w-\alpha'-\beta')>0$.

$$\int_0^{\pi/2} e^{i(2w+1)\pi\theta}(\sin\theta)^{w-1}(\cos\theta)^{w-1} {}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'}{2};e^{i\theta}\cos\theta\right]d\theta$$

$$=\frac{e^{i\pi(w-1)/2}\Gamma(w-1)\Gamma\left(w-\frac{\alpha'-\beta'}{2}-1\right)\Gamma\left(\frac{\alpha'+\beta'}{2}\right)}{2^{2w-\alpha'-\beta'}\Gamma(\alpha')\Gamma(\beta')}$$

$$.\left[\frac{(2w-\alpha'-\beta'-2)\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)}{\Gamma\left(w-\frac{\alpha'}{2}\right)\Gamma\left(w-\frac{\beta'+1}{2}\right)}+\frac{(2w+\alpha'-\beta'-2)\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)}{\Gamma\left(w-\frac{\beta'}{2}\right)\Gamma\left(w-\frac{\alpha'+1}{2}\right)}\right] \qquad (2.4)$$

Where $\mathrm{Re}(w)>0, \mathrm{Re}(2w-\alpha'-\beta')>0$.

III. Main Integrals

We shall evaluate the following four finite double integrals involving hypergeometric functions and $\overline{H}$ -function.

First Integral

$$\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho-1}\,{}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$

$$.e^{i\pi(2w+1)\theta}(\sin\theta)^w(\cos\theta)^w\,{}_2F_1\left[\alpha',\beta';\frac{(\alpha'+\beta'+2)}{2};e^{i\theta}\cos\theta\right]$$

$$.\ \overline{H}{}^{m,n}_{p,q}\left[zx^{\rho}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2w_1 i\theta}(\sin\theta)^{w_1}(\cos\theta)^{w_1}\Big|{}^{A^*}_{B^*}\right]d\theta dx$$

$$=\frac{2^{\alpha+\beta-2\rho-2}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\overline{H}_1-\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\overline{H}_2\right]$$

$$\frac{e^{i\pi(w+1)/2}\Gamma\left(\frac{\alpha'+\beta'+2}{2}\right)}{2^{2w-\alpha'-\beta'+2}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)\overline{H}_3-\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)\overline{H}_4\right] \qquad (3.1)$$

Where $\overline{H}_1, \overline{H}_2, \overline{H}_3$ and $\overline{H}_4$ are given as follows:

$$\overline{H}_1=\overline{H}{}^{m,n+3}_{p+3,q+3}\left[\frac{z2^{-2\rho_1-2}}{(1+a)^{\rho_1}(1+b)^{\rho_1}}\left|\begin{matrix}(\alpha-\beta-2\rho;2\rho_1;1),\left(1+\frac{\alpha+\beta}{2}-\rho;\rho_1;1\right),(1-\rho;\rho_1;1),A^*\\ B^*,(1-2\rho+\alpha+\beta;2\rho_1;1),\left(\frac{\alpha}{2}-\rho;\rho_1;1\right),\left(\frac{\beta+1}{2}-\rho;\rho_1;1\right)\end{matrix}\right.\right] \qquad (3.2)$$

$$\overline{H}_2=\overline{H}{}^{m,n+3}_{p+3,q+3}\left[\frac{z2^{-2\rho_1-2}}{(1+a)^{\rho_1}(1+b)^{\rho_1}}\left|\begin{matrix}(-\alpha+\beta-2\rho;2\rho_1;1),\left(1+\frac{\alpha+\beta}{2}-\rho;\rho_1;1\right),(1-\rho;\rho_1;1),A^*\\ B^*,(1-2\rho-\alpha+\beta;2\rho_1;1),\left(\frac{\alpha+1}{2}-\rho;\rho_1;1\right),\left(\frac{\beta}{2}-\rho;\rho_1;1\right)\end{matrix}\right.\right] \qquad (3.3)$$

$$\overline{H}_3=\overline{H}{}^{m,n+3}_{p+3,q+3}\left[\frac{ze^{i\pi w_1/2}}{2^{2w_1}}\left|\begin{matrix}(\alpha'-\beta'-2w;2w_1;1),\left(1+\frac{\alpha'+\beta'}{2}-w;w_1;1\right),(1-w;w_1;1),A^*\\ B^*,(1-2w+\alpha'+\beta';2w_1;1),\left(\frac{\alpha'}{2}-w;w_1;1\right),\left(\frac{\beta'+1}{2}-w;w_1;1\right)\end{matrix}\right.\right] \qquad (3.4)$$

$$\overline{H}_4=\overline{H}{}^{m,n+3}_{p+3,q+3}\left[\frac{ze^{i\pi w_1/2}}{2^{2w_1}}\left|\begin{matrix}(-\alpha'+\beta'-2w;2w_1;1),\left(1+\frac{\alpha'+\beta'}{2}-w;w_1;1\right),(1-w;w_1;1),A^*\\ B^*,(1-2w-\alpha'+\beta';2w_1;1),\left(\frac{\alpha'+1}{2}-w;w_1;1\right),\left(\frac{\beta'}{2}-w;w_1;1\right)\end{matrix}\right.\right] \qquad (3.5)$$

(i) $\mathrm{Re}(\rho)>0, \mathrm{Re}(w)>0, |\arg z|<\frac{\theta\pi}{2}, \theta>0,$

(ii) $\mathrm{Re}(2\rho-\alpha-\beta+2\rho_1 B_j\frac{b_j}{\beta_j})>0, \left(B_j\frac{b_j}{\beta_j}\right)>0,$

(iii) $\mathrm{Re}\left(2w-\alpha'-\beta'+2w_1 B_j\frac{b_j}{\beta_j}\right)>0, \left(B_j\frac{b_j}{\beta_j}\right)>0.$

Second Integral

$$\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho+1}\,{}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$
$$e^{i\pi(2w-1)\theta}(\sin\theta)^{w-2}(\cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{(\alpha'+\beta')}{2};e^{i\theta}\cos\theta\right]$$
$$\overline{H}^{m,n}_{p,q}\left[zx^{\rho}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2w_1 i\theta}(\sin\theta)^{w_1}(\cos\theta)^{w_1}\left|{}^{A^*}_{B^*}\right.\right]d\theta dx$$
$$=\frac{2^{\alpha+\beta-2\rho-1}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\overline{H}_5-\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+\alpha}{2}\right)\overline{H}_6\right]$$
$$\frac{e^{i\pi(w+1)/2}\Gamma\left(\frac{\alpha'+\beta'+2}{2}\right)}{2^{2w-\beta'+1}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)\overline{H}_7-\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+2}{2}\right)\overline{H}_8\right] \quad (3.6)$$

Where $\overline{H}_5, \overline{H}_6, \overline{H}_7$ and $\overline{H}_8$ are given as follows:

$$\overline{H}_5=\overline{H}^{m,n+3}_{p+3,q+3}\left[\frac{z2^{-2\rho_1}}{(1+a)^{\rho_1}(1+b)^{\rho_1}}\left|\begin{matrix}(2+\alpha-\beta-2\rho;2\rho_1;1),\left(2+\frac{\alpha+\beta}{2}-\rho;\rho_1;1\right),(2-\rho;\rho_1;1),A^*\\ B^*,(3-2\rho+\alpha+\beta;2\rho_1;1),\left(1+\frac{\alpha}{2}-\rho;\rho_1;1\right),\left(\frac{\beta+3}{2}-\rho;\rho_1;1\right)\end{matrix}\right.\right] \quad (3.7)$$

$$\overline{H}_6=\overline{H}^{m,n+3}_{p+3,q+3}\left[\frac{z2^{-2\rho_1}}{(1+a)^{\rho_1}(1+b)^{\rho_1}}\left|\begin{matrix}(2-\alpha+\beta-2\rho;2\rho_1;1),\left(2+\frac{\alpha+\beta}{2}-\rho;\rho_1;1\right),(2-\rho;\rho_1;1),A^*\\ B^*,(3-2\rho-\alpha+\beta;2\rho_1;1),\left(\frac{\alpha+3}{2}-\rho;\rho_1;1\right),\left(1+\frac{\beta}{2}-\rho;\rho_1;1\right)\end{matrix}\right.\right] \quad (3.8)$$

$$\overline{H}_7=\overline{H}^{m,n+3}_{p+3,q+3}\left[\frac{ze^{i\pi w_1/2}}{2^{2w_1}}\left|\begin{matrix}(2+\alpha'-\beta'-2w;2w_1;1),\left(2+\frac{\alpha'+\beta'}{2}-w;w_1;1\right),(2-w;w_1;1),A^*\\ B^*,(3-2w+\alpha'+\beta';2w_1;1),\left(1+\frac{\alpha'}{2}-w;w_1;1\right),\left(\frac{\beta'+3}{2}-w;w_1;1\right)\end{matrix}\right.\right] \quad (3.9)$$

$$\overline{H}_8=\overline{H}^{m,n+3}_{p+3,q+3}\left[\frac{ze^{i\pi w_1/2}}{2^{2w_1}}\left|\begin{matrix}(2-\alpha'+\beta'-2w;2w_1;1),\left(2+\frac{\alpha'+\beta'}{2}-w;w_1;1\right),(2-w;w_1;1),A^*\\ B^*,(3-2w-\alpha'+\beta';2w_1;1),\left(\frac{\alpha'+3}{2}-w;w_1;1\right),\left(1+\frac{\beta'}{2}-w;w_1;1\right)\end{matrix}\right.\right] \quad (3.10)$$

The integral (3.6) is valid if the following set of (sufficient) conditions are satisfied:

(i) $\mathrm{Re}(\rho)>1, \mathrm{Re}(w)>1, |\arg z|<\frac{\theta\pi}{2}, \theta>0,$

(ii) $\mathrm{Re}(2\rho-\alpha-\beta+2\rho_1 B_j\frac{b_j}{\beta_j})>2, \mathrm{Re}\left(B_j\frac{b_j}{\beta_j}\right)>0,$

(iii) $\mathrm{Re}\left(2w-\alpha'-\beta'+2w_1 B_j\frac{b_j}{\beta_j}\right)>2, \left(B_j\frac{b_j}{\beta_j}\right)>0.$

Third Integral

$$\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho+1}\,{}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$
$$e^{i\pi(2w-1)\theta}(\sin\theta)^{w-1}(\cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{(\alpha'+\beta')}{2};e^{i\theta}\cos\theta\right]$$
$$\overline{H}^{m,n}_{p,q}\left[zx^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2w_1 i\theta}(\sin\theta)^{w_1}(\cos\theta)^{w_1}\left|{}^{A^*}_{B^*}\right.\right]d\theta dx$$
$$=\frac{2^{\alpha+\beta-2\rho-2}}{\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\overline{H}_1-\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\overline{H}_2\right]$$

$$\frac{e^{i\pi(w-1)/2}\Gamma\left(\frac{\alpha'+\beta'}{2}\right)}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)\overline{H}_7-\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)\overline{H}_8\right] \qquad (3.11)$$

Where $\overline{H}_1, \overline{H}_2, \overline{H}_7$ and $\overline{H}_8$ are mentioned in (3.2), (3.3), (3.9) and (3.10) respectively and set of conditions are as follows:

(i) $\mathrm{Re}(\rho)>0, \mathrm{Re}(w)>1, |\arg z|<\frac{\theta\pi}{2}, \theta>0,$

(ii) $\mathrm{Re}(2\rho-\alpha-\beta+2\rho_1 B_j\frac{b_j}{\beta_j})>0, \left(B_j\frac{b_j}{\beta_j}\right)>0,$

(iii) $\mathrm{Re}\left(2w-\alpha'-\beta'+2w_1 B_j\frac{b_j}{\beta_j}\right)>0, \left(B_j\frac{b_j}{\beta_j}\right)>0.$

Fourth Integral

$$\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho-2}[1+ax+b(1-x)]^{-2\rho+1}\,{}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$

$$e^{i\pi(2w+1)\theta}(\sin\theta)^w(\cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{(\alpha'+\beta'+2)}{2};e^{i\theta}\cos\theta\right]$$

$$\overline{H}_{p,q}^{m,n}\left[zx^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2w_1 i\theta}(\sin\theta)^{w_1}(\cos\theta)^{w_1}\Big|_{B^*}^{A^*}\right]d\theta dx$$

$$=\frac{2^{\alpha+\beta-2\rho-1}\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\overline{H}_5-\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\overline{H}_6\right]$$

$$\frac{e^{i\pi(w+1)/2}\Gamma\left(\frac{\alpha'+\beta'+2}{2}\right)}{2^{2w+\alpha'-\beta'+2}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)\overline{H}_3-\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)\overline{H}_4\right] \qquad (3.12)$$

Where $\overline{H}_5, \overline{H}_6, \overline{H}_3$ and $\overline{H}_4$ are mentioned in (3.4), (3.5), (3.7) and (3.8) respectively and set of conditions are as follows:

(i) $\mathrm{Re}(\rho)>0, \mathrm{Re}(w)>1, |\arg z|<\frac{\theta\pi}{2}, \theta>0,$

(ii) $\mathrm{Re}(2\rho-\alpha-\beta+2\rho_1 B_j\frac{b_j}{\beta_j})>0, \left(B_j\frac{b_j}{\beta_j}\right)>0,$

(iii) $\mathrm{Re}\left(2w-\alpha'-\beta'+2w_1 B_j\frac{b_j}{\beta_j}\right)>2, \left(B_j\frac{b_j}{\beta_j}\right)>0.$

Proof: To establish (3.1), express the $\overline{H}$-function on the left hand side using (1.1)in single Mellin-Barnes contour integrals and interchanging the order of integration which is justifiable due to absolute convergence of the integrals, we have:

$$=\frac{1}{2\pi w}\int_L\frac{\prod_{j=1}^{n}\left\{\Gamma\left(1-a_j+\alpha_j s\right)\right\}^{A_j}\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right)}{\prod_{j=m+1}^{q}\left\{\Gamma\left(1-b_j+\beta_j s\right)\right\}^{B_j}\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j s\right)}\int_0^1 x^{\rho+\rho_1 s-1}(1-x)^{\rho+\rho_1 s}$$

$$[1+ax+b(1-x)]^{-2\rho-2\rho_1 s-1}\,{}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]dx$$

$$\int_0^{\pi/2} e^{i(2w+2w_1s+1)}(\sin\theta)^{w+w_1s}(\cos\theta)^{w+w_1s}\,{}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}\cos\theta\right]d\theta$$

Evaluate the inner integral with the help of (2.1) and (2.3) and then applying (1.1), we get R.H.S of (3.1)in terms of product of $\overline{H}$ -function.

The proof of second, third and fourth integrals are similar to the first with the only difference that here we make use of known integrals, (2.1), (2.4) for second result (2.1) and (2.4) for third result and (2.2) and (2.3) for fourth result respectively instead of (2.1) and (2.3).

IV. Special Cases

(i) Putting $a=b$ in (3.6), we get

$$\int_0^1\int_0^{\pi/2} x^{\rho-1}(1-x)^{\rho}[1+b]^{-2\rho+1}\,{}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};x\right]$$

$$e^{i\pi(2w+1)\theta}(\sin\theta)^{w-2}(\cos\theta)^{w-1}\,{}_2F_1\left[\alpha',\beta';\frac{(\alpha'+\beta'+2)}{2};e^{i\theta}\cos\theta\right]$$

$$\overline{H}^{m,n}_{p,q}\left[zx^{\rho_1}(1-x)^{\rho_1}[1+b]^{-2\rho_1}e^{2w_1i\theta}(\sin\theta)^{w_1}(\cos\theta)^{w_1}\Big|^{A*}_{B*}\right]d\theta dx$$

$$=\frac{2^{\alpha+\beta-2\rho-1}\Gamma\left(\frac{\alpha+\beta}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)(1+b)^{2\rho}}\left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\overline{H}_5-\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\overline{H}_6\right]$$

$$\frac{e^{i\pi(w-1)/2}\Gamma\left(\frac{\alpha'+\beta'}{2}\right)}{2^{2w+\alpha'-\beta'+1}\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)\overline{H}_7-\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+2}{2}\right)\overline{H}_8\right] \quad (4.1)$$

Where $\overline{H}_5,\overline{H}_6,\overline{H}_7$ and $\overline{H}_8$ are integrals mentioned in the second integral are satisfied and the set of conditions mentioned with second integrals are also satisfied.

(ii) If we put $A_j=B_j=1$ in (3.1), the result will be changed into the Fox's H -function instead of the $\overline{H}$ -function.

V. Application

We shall define the Riemann-Liouville fractional derivative of function $f(x)$ of order σ (or alternatively, σ^{th} order fractional integral) ([7], p.181;,[8],p.49) by

$${}_aD_x^{\sigma}\{f(x)\}=\begin{cases}\frac{1}{\Gamma(-\sigma)}\int_a^x (x-t)^{-\sigma-1}f(t)dt, \mathrm{Re}(\sigma)<0\\ \frac{d^q}{dx^q}\,{}_aD_x^{\sigma-q}\{f(x)\},\ q-1\le \mathrm{Re}(\sigma)<q\end{cases}, \quad (5.1)$$

Where q is a positive integer and the integral exists. For simplicity the special case of the fractional derivative operator ${}_aD_x^{\sigma}$, when $a=0$ will be written as D_x^{σ}. Thus we have

$$D_x^{\sigma}={}_0D_x^{\sigma} \quad (5.2)$$

The preliminary result (3.1), after interchanging the order of integration which is justified due to absolute convergence of the integrals can be rewritten as the following fractional integral formula:

$$\int_0^{\pi/2} e^{i\pi(2w+1)\theta}(\sin\theta)^{w}(\cos\theta)^{w}\,{}_2F_1\left[\alpha',\beta';\frac{\alpha'+\beta'+2}{2};e^{i\theta}\cos\theta\right]$$

$$. D_x^{-\rho-1}\left\{\begin{matrix}\left\{x^{\rho-1}[1+ax+b(1-x)]^{-2\rho-1}\right\}{}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta+2}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right] \\ \overline{H}_{p,q}^{m,n}\left[zx^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1}e^{2w_1i\theta}(\sin\theta)^{w_1}(\cos\theta)^{w_1}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]\end{matrix}\right\}d\theta$$

Using (5.1) and evaluating the integral with the help of (2.3), we get

$$=\frac{2^{\alpha+\beta-2\rho-2}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}{\Gamma(\rho-1)\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}}\left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\overline{H}_1-\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)\overline{H}_2\right]$$

$$\frac{e^{i\pi(w+1)/2}\Gamma\left(\frac{\alpha'+\beta'+2}{2}\right)}{2^{2w-\alpha'-\beta'+2}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')}\left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)\overline{H}_3-\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)\overline{H}_4\right] \quad (5.3)$$

Where $\mathrm{Re}(-\rho-1)>0$ and all the conditions of validity mentioned with (3.1) are satisfied. The fractional integral formula given by (5.3) is also quite general in nature and can easily yield Riemann-Liouville fractional integrals of a large number of simpler functions and polynomials merely by specializing the parameters $\overline{H}$ occurring in it which may find applications in electromagnetic theory and probability.

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