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On πgr-Continuous functions.

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Abstract:
The aim of this paper is to consider and characterize πgr-closure ,πgr- interior, πgr-continuous and almost πgr-continuous functions and to relate these concepts to the classes of πgr-compact spaces, πgr-connected spaces.

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Key Words: πgr-cl(A), πgr-int(A), almost πgr-continuous, πgr-compactness, πgr-connectedness and $\tau^*_{\pi gr}$.

1.Introduction

Levine [10] initiated the study of so-called generalized closed set(briefly g-closed set).The notion has been studied extensively in recent years by many topologists, because g-closed sets are not only the generalization of closed sets. More importantly, they also suggested several new properties of topological spaces. Later on N.Palaniappan[11,12] studied the concept of regular generalized closed set in topological space. Zaitsev [16]introduced the concept of π-closed sets in topological space. Dontchev.J and Noiri.T[4] introduced the concept of πg-closed set in topological space.

Hussain(1966) [7] ,M.K.Singal and A.R.Singal(1968)[14] introduced the concept of almost continuity in topological spaces. K.Balachandran , P.Sundram and Maki [2] introduced a class of compactness called GO-compact spaces and GO-connected spaces using g-open cover.

Recently Jeyanthi.Vand Janaki.C [9] introduced and studied the properties of πgr-closed sets in topological spaces. The purpose of this paper is to study πgr-closure, πgr -interior, almost πgr-continuous functions and some of its basic properties . Further , we introduce the concepts of πgr-compact spaces , πgr-connected spaces and study their behaviours under πgr-continuous functions**.**

2. Preliminaries

Throughout this paper, X and Y denote the topological spaces (X,τ) and (Y,σ) respectively, on which no separation axioms are assumed.

Let us recall the following definitions which are useful in the sequal.

Definition :2.1

A subset A of a topological space X is said to be
(i) a regular open[12] if A = int(cl(A)) and regular closed if A = cl(int(A))
(ii) π- open [16] if A is the finite union of regular open sets and the complement of π- open is π- closed set in X.

The family of all open sets [regular open, π-open] sets of X will be denoted by O(X)(resp. RO(X), πO(X)]

Definition :2.2

A subset A of topological space X is said to be
(1) a generalized closed set [10] (g-closed set) if cl (A) ⊂ U whenever A⊂ U and U ∈ O(X).
(2) a regular generalized closed [12] (briefly rg-closed set) if cl(A)⊂U whenever A⊂ U and U∈ RO(X).
(3) a generalized pre regular closed set [5] (briefly gpr -closed set) if pcl (A) ⊂U whenever A ⊂ U and U ∈ RO(X) .
(4) a π-generalized closed [4] (briefly πg- closed set) if cl (A) ⊂ U whenever A ⊂ U and U ∈ πO(X).
(5) a πgα - closed set [8]if αcl (A) ⊂ U whenever A⊂U and U∈ πO(X).
(6)a π*g-closed set [6]if cl(int(A)) ⊂ U whenever A⊂U and U∈ πO(X).
(7) a πgb-closed set[15] if bcl(A) ⊂ U whenever A⊂U and U∈ πO(X).
(8) a πgp-closed set [13]if pcl(A) ⊂ U whenever A⊂ U and U∈ πO(X).

(9) a πgs-closed set[1] if scl(A) ⊂ U whenever A⊂U and U∈ πO(X).
(10) a generalized regular closed set [12] (briefly g-r-closed set) if rcl (A) ⊂U whenever A ⊂ U and U ∈ Open in X.
(11) A subset A of X is called πgr- closed set[9] in X if rcl(A) ⊂ U whenever A⊂ U and U is π-open in X.The complement of πgr -closed set is πgr-open set.
We denote the family of all πgr-closed (resp. πgr-open)sets in X by πGRC(X)(resp. πGRO(X)).

Definition :2.3

A map f: X→Y is said to be
(1) continuous function[10] if $f^{-1}(V)$ is closed in X for every closed set V in Y.
(2) regular continuous [12] if $f^{-1}(V)$ is regular closed in X for every closed set V in Y.
(3) πgr- continuous[9] if $f^{-1}(V)$ is πgr- closed in X for every closed set V of Y.
(4) almost continuous [14]if $f^{-1}(V)$ is closed in X for every regular closed set V of Y
(5) almost π-continuous[4]if $f^{-1}(V)$ is π-closed in X for every regular closed set V in Y.
(6) almost πgb-continuous[15]if $f^{-1}(V)$ is πgb-closed in X for every regular closed set V in Y.
(7) almost πgα-continuous[8]if $f^{-1}(V)$ is πgα-closed in X for every regular closed set V in Y.
(8) almost πg-continuous[4]if $f^{-1}(V)$ is πg-closed in X for every regular closed set V in Y.
(9) almost π*g-continuous[6]if $f^{-1}(V)$ is π*g-closed in X for every regular closed set V in Y.
(10)almost gpr-continuous[5]if $f^{-1}(V)$ is gpr-closed in X for every regular closed set V in Y.
(11) pre-regular closed[12]if f(F) is regular closed in Y for every regular closed set F in X.

Definition: 2.4 .

Regular closure (briefly r-closure) [12]of a set A is defined as the intersection of all regular closed sets containing the set and regular interior(briefly r-interior)[12] of a set A is the union of regular open set contained in the set.

The above are denoted by rcl(A) and r int(A)

Definition:2.5

A map f: X→Y is said to be
(1) a irresolute function[1] if $f^{-1}(V)$ is closed in X for every closed set V in Y.
(2) an R-map [3]if $f^{-1}(V)$ is regular-closed in X for every regular closed set V in Y.
(3) πgr- irresolute [9] if $f^{-1}(V)$ is πgr- closed in X for every πgr -closed set V of Y.

3. πgr –Closure and Interior

Definition: 3.1

For any set A⊂X, the πgr-closure of A is defined as the intersection of πgr -closed sets containing A. The complement of πgr-closure is πgr-interior.

We write πgr-cl(A) = ∩{ F:A⊂F is πgr-closed in X}

Lemma: 3.2

For an x∈X, x∈ πgr-cl(A) iff V∩A ≠φ for every πgr-open set V containing x.

Proof:

First, let us suppose that there exists a πgr-open set V containing x such that V∩A= φ.
Since A ⊂ X–V, πgr-cl(A) ⊂ X– V
⇒ x ∉ πgr-cl(A), which is a contradiction to the fact that x∈ πgr-cl(A). Hence V∩A ≠φ for every πgr-open set V containing x.

On the other hand, let x ∉ πgr-cl(A). Then there exists a πgr-closed subset F containing A such that x∉F. Then x∈X–F and X-F is πgr-open. Also, (X–F) ∩A ≠φ, a contradiction. Hence the lemma.

Lemma :3.3

Let A and B be subsets of (X, τ) . Then
(i) πgr-cl (φ) = φ and πgr-cl(X) =X.
(ii) If A ⊂B, then πgr-cl(A) ⊂ πgr-cl(B)
(iii) A ⊂ πgr-cl(A)
(iv) πgr-cl(A)= πgr-cl(πgr-cl(A))
(v) πgr-cl(A∪B)= πgr-cl(A) ∪ πgr-cl(B)

Proof: Obvious.

Remark: 3.4

If $A \subset X$ is πgr-closed, then πgr-cl(A)=A.But the converse is need not be true as seen in the following example.

Example: 3.5

Let X = { a,b,c,d}, τ = {φ, X, {a}, {b} , {a,b} , {a,b,c}}.
Let A = {a}, πgr-cl(A) = {a} =A , but A = {a} is not πgr-closed in X.

Lemma: 3.6

Let A and B be subsets of X. Then πgr-cl(A∩B) ⊂πgr-cl(A) ∩ πgr-cl(B)

Proof:

Since A∩B ⊂ A, B.

⟹ πgr-cl(A∩B) ⊂ πgr-cl(A), πgr-cl(A∩B) ⊂ πgr-cl(B)

⟹ πgr-cl(A∩B) ⊂ πgr-cl(A)∩ πgr-cl(B)

Remark:3.7

The converse of the above need not be true as seen in the following example.

Example:3.8

Let X = { a,b,c,d,e}, τ = {φ, X, {a}, {e} , {a,e} , {c,d},{a,c,d},{a,c,d,e},{c,d,e}}
Let A = {a,c,e}⊂X,B={d}⊂X. Then πgr-cl(A) = {a,b,c,e}, πgr-cl(B) = {b,d}
But πgr-cl(A)∩ πgr-cl(B)= {b} ⊄ πgr-cl(A∩B).

Remark: 3.9

We denote πgr-closure of A by πGRCL(X) ,πgr-closed sets in a topological space by πgrC(X) , πgr-open sets by πgrO(X).

Definition:3.10

$\tau^*_{\pi gr}$ = { V ⊂X/ πgr-cl(X–V) = X–V}

Theorem:3.11

If πgrO(X) is a topology, then $\tau^*_{\pi gr}$ is a topology.

Proof:

Clearly, φ, X ∈ $\tau^*_{\pi gr}$. Let $\{A_i : i \in A\} \in \tau^*_{\pi gr}$.

πgr-cl(X– (∪ A_i)) = πgr-cl(∩(X–A_i))
⊂∩πgr-cl (X–A_i)
=∩ (X–A_i)
= X–∪ A_i

Hence ∪ A_i ∈ $\tau^*_{\pi gr}$.

Let A, B ∈ $\tau^*_{\pi gr}$.

Now, πgr-cl(X– (A∩B)= πgr-cl((X–A) ∪ (X–B))
= πgr-cl(X–A)∪ πgr-cl((X–B)
= (X–A) ∪ (X–B)

Thus A∩B∈ $\tau^*_{\pi gr}$ and hence $\tau^*_{\pi gr}$ is a topology.

Definition: 3.12

Let X be a topological space and let x∈X. A subset N of X is said to be πgr-nbhd of x if there exists a πgr-open set G such that x∈G ⊂ N.

Definition :3.13

Let A be a subset of X. A point x∈ A is said to be πgr-interior point of A if A is a πgr-nbhd of x. The set of all πgr-interior of A and is denoted by πgr-int(A)

Theorem: 3.1 4

If A be a subset of X. Then πgr-int(A)=∪{G: G is πgr-open, G⊂A}

Proof: Straight forward.

Theorem:3.15

Let A and B be subsets of X. Then

(i) πgr-int(X) =X, πgr-int(φ) = φ
(ii) πgr-int(A) ⊂A
(iii) If B is any πgr-open set contained in A, then B ⊂ πgr-int(A)
(iv) If A ⊂B , then πgr-int(A) ⊂ πgr-int(B)
(v) πgr-int(πgr-int(A))= πgr-int(A)

Proof:

Straight Forward.

Theorem:3.16

If a subset A of a space X is πgr-open, then πgr-int(A) =A.

Proof: Obvious.

Remark:3.17

The converse of the above need not be true as seen in the following example.

Example:3.18

Let X = { a,b,c,d}, τ = {φ, X, {a}, {b} , {a,b} , {a,b,c}}.
Let A = {c,d} . Then πgr-int(A) = {c,d} =A . But A={c,d} is not πgr-open.

Theorem:3.19

If A and B are subsets of X, then πgr-int (A) ∪ πgr-int(B) ⊂ πgr-int (A∪B)

Proof:

We know that A ⊂ A∪B and B ⊂ A∪B
Then πgr-int(A) ⊂ πgr-int (A∪B), πgr-int(B) ⊂ πgr-int (A∪B)
Hence πgr-int (A) ∪ πgr-int(B) ⊂ πgr-int (A∪B).

Theorem:3.20

If A and B are subsets of a space X, then πgr-int (A∩B)= πgr-int (A)∩ πgr-int (B)

Proof:

We know that A∩B⊂A, A∩B⊂B.Then πgr-int (A∩B) ⊂ πgr-int(A) and πgr-int (A∩B) ⊂ πgr-int(B).
⟹ πgr-int (A∩B)= πgr-int (A)∩ πgr-int (B)-------(1)
Again , let x∈ πgr-int (A)∩ πgr-int (B). Then x∈ πgr-int (A)and x∈πgr-int (B).Hence x is a πgr-interior point of each of sets A and B. It follows that A and B are πgr-nbhds of x, so that their intersection A∩B is also a πgr-nbhd of x . Hence x∈ πgr-int (A∩B)

Thus, x∈ πgr-int (A)∩ πgr-int (B) ⟹ x∈ πgr-int (A∩B)
Therefore, πgr-int (A)∩ πgr-int (B) ⊂ πgr-int (A∩B)-----------(2)
From (1) and (2) , πgr-int (A∩B)= πgr-int (A)∩ πgr-int (B).

Theorem: 3.21

If A is a subset of X, then
(i) r-int(A) ⊂πgr-int(A) and
(ii) (X– πgr-int(A)) = πgr-cl(X–A) and (X– πgr-cl(A)) = πgr-int(X–A).

Proof:

Straight forward.

4. πgr-continuous functions and Almost πgr-continuous functions.

Theorem:4.1

Let X be a πgr-$T_{1/2}$-space and f:X→Y be a function. Then f is πgr-continuous iff f is regular continuous.

Proof:

Let f be a πgr-continuous function. Then $f^{-1}(V)$ is πgr-closed in X for every closed set V in Y. Since X is a πgr-$T_{1/2}$-space, every πgr-closed set is regular closed. Hence $f^{-1}(V)$ is regular closed in X for every closed set V in Y and hence f is regular continuous.

Let f be a regular continuous function in X.Then $f^{-1}(V)$ is regular closed in X for every closed set V in Y. Since every regular closed set is πgr-closed. Then $f^{-1}(V)$ is πgr-closed in X for every closed set V in Y and hence f is πgr-closed.

Theorem:4.2

Let f: (X,τ) →(Y, σ) be a function, then the following are equivalent.

a)f is πgr-continuous

b) The inverse image of every open set in Y is πgr-open in X.

Proof:

Follows from the definitions.

Theorem:4.3

If f: (X,τ) →(Y, σ) is πgr-continuous, then f(πgr-cl(A)) ⊂cl(f(A)) for every subset A of X.

Proof:

Let A⊂X. Since f is πgr-continuous and A ⊂f^{-1}(cl(f(A))), we obtain πgr-cl(A)⊂f^{-1}(cl(f(A)) and then f((πgr-cl(A)) ⊂cl(f(A))

Remark:4.4

The converse of the above need not be true as seen in the following example.

Example:4.5

Let X = { a,b,c,d}, τ = {φ, X, {a}, {b} , {a,b}}. σ={φ,Y,{c,d}}. Let f: (X,τ) →(Y, σ) be an identity map.Let A = {a,b} . Then πgr-cl({a,b}) = {a,b}⊂f^{-1}(cl(f({a,b}))) =X.

But f^{-1}({a,b}) = {a,b} is not πgr-closed in X.Hence f is not πgr-continuous.

Proposition:4.6

Let f: (X,τ) →(Y, σ) be a πgr-continuous function and H be π-open, πgr-closed subset of X. Assume that πgrC(X, τ) closed under finite intersections. Then the restriction be f/H: (H, τ/H) →(Y, σ) is πgr-continuous.

Proof:

Let F be any regular closed subset in Y. By hypothesis and our assumption f^{-1} (F) ∩H_1, it is πgr-closed in X.Since $(f/H)^{-1}$ (F) = H_1,it is sufficient to show that H_1 is πgr-closed in H.Let H_1 ⊂ G_1, where G_1 is any π-open set in H.We know that a subset A of X is open , then πO(A, τ/A)= {V∩A: V∈ πO(X,τ)}-------(1).By (1), G_1= G∩H for some π-open set G in X.

Then H_1 ⊂G_1⊂G and H_1 is πgr-closed in X implies $rcl_x(H_1)$ = $rcl_x(H_1)$ ∩H ⊂G∩H= G_1 and so H_1 is πgr-closed in H. Therefore, f/H is πgr-continuous.

Generalization of Pasting Lemma for πgr-continuous maps.

Theorem:4.7

Let X = G∪H be a topological space with topology τ and Y be a topological space with topology σ. Let f: (G, τ/G) →(Y, σ) and g: (H, τ/H) →(Y, σ) be πgr-continuous functions such that f(x)=g(x) for every x ∈ G∩H. Suppose that both G and H are π-open and πgr-closed in X. Then their combination (f∇ g): (X, τ) →(Y, σ) defined by (f∇ g)(x)=f(x) if x∈G and (f∇ g)(x) =g(x) if x∈H is πgr-continuous.

Proof:

Let F be any closed set in Y.Clearly $(f\nabla g)^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$. Since f^{-1}(F) is πgr-closed in G and G is π-open in X and πgr-closed in X, f^{-1}(F) is πgr-closed in X.Similarly, g^{-1}(F) is πgr-closed in X. Therefore, (f∇ g) is πgr-continuous.

Proposition :4.8

If a function f: (X, τ) →(Y, σ) is πgr-irresolute, then

(i) f(πgr-cl(A)) ⊂πgr-cl (f(A)) for every subset A of X.

(ii) πgr-cl(f^{-1}(B)) ⊂f^{-1}(πgr-cl (B))for every subset B of Y.

Proof:

(i)For every A⊂X, πgr-cl(f(A)) is πgr-closed in Y.By hypothesis, f^{-1}(πgr-cl(f(A)) is πgr-closed in X. Also,A⊂f^{-1}(f(A)) ⊂f^{-1}(πgr-cl(A)). By the definition of πgr-closure , we have πgr-cl(A) ⊂f^{-1}(πgr-cl(A)).Hence , we get f(πgr-cl(A)) ⊂ πgr-cl(f(A))

(ii)πgr-cl(B) is πgr-closed in Y and so by hypothesis , f^{-1}(πgr-cl(B)) is πgr-closed in X. Since f^{-1}(B) ⊂f^{-1}(πgr-cl(B)),it follows that πgr-cl(f^{-1}(B)) ⊂f^{-1}(πgr-cl(B)).

Definition:4.9

A function f: (X,τ) →(Y, σ) is called almost-πgr- continuous if f^{-1}(V) is πgr- closed in X for every regular closed set V of Y.

Theorem:4.10

For a function f:X→Y, the following are equivalent to one another.

(i) f is almost πgr-continuous.
(ii) f^{-1}(V) is πgr-open in X for every regular open set V of Y.
(iii) f^{-1}(int-cl(V)) is πgr-open in X for every open set V of Y.
(iv) f^{-1}(cl-int(V)) is πgr-closed in X for every closed set V of Y.

Proof:(i) ⇒ (ii)

Let V be a regular open subset of Y. Since Y–V is regular closed and f is almost πgr-continuous, then f^{-1}(Y–V) = X- f^{-1}(V) is πgr-closed in X. Thus f^{-1}(V) is πgr-open on X.

(ii) ⇒ (i)

Let V be a regular closed subset of Y . Then Y–V is regular open.By hypothesis, f^{-1}(Y–V)=X–f^{-1}(V) is πgr-open in X. Then f^{-1}(V) is πgr-closed and hence f is almost πgr-continuous.

(ii) ⇒ (iii)

Let V be an open subset of Y. Then int(cl(V)) is regular open . By hypothesis f^{-1}(int(cl(V))) is πgr-open in X.

(iii) ⇒ (ii)

Let V be a regular open subset of Y. Since V– int(cl(V))) and every regular open set is open, then f^{-1}(V) is πgr-open in X.

(iii) ⇒ (iv)

Let V be a closed subset of Y. Then Y-V is open. By hypothesis,
f^{-1}(int(cl(Y–V)))= f^{-1}(Y–cl(int(V)))
= X–f^{-1}(cl(int (V))) is πgr-open in X.
Hence f^{-1}(cl(int(V))) is πgr-closed in X.

(iv) ⇒ (iii)

Let V be a open subset of Y. Then Y–V is closed. By hypothesis,
f^{-1}(cl(int(Y–V)))= f^{-1}(Y–int(cl(V)))
= X–f^{-1}(int(cl(V))) is πgr-closed in X. Hence f^{-1}(int(cl(V))) is πgr-open in X.

Remark:4.11

Every πgr-continuous function is almost πgr-continuous but not conversely.

Example:4.12

Let X = { a,b,c,d}=Y, τ = {φ, X, {a}, {b} , {a,b}},σ={φ, Y,{a},{d},{a,d},{a,b,d}},
Let f:X→Y be defined by f(a)=c, f(b)=a,f(c)=d and f(d)=b.Here f is almost πgr-continuous but nor πgr-continuous.

Remark:4.13

An R-map is almost πgr-continuous

Proof: Follows from the definitions.

Remark :4.14

The converse of the above need not be true as seen in the following example.

Example:4.15

Let X = { a,b,c,d}=Y, τ = {φ, X, {a}, {b} , {a,b}},σ={φ, Y,{a},{d},{a,d},{a,b,d}}, Let f:X→Y be an identity map .Here f is almost πgr-continuous but not an R-map.

Theorem:4.16

Let X be a πgr-$T_{1/2}$- space. Then f:X→Y is almost πgr-continuous iff f is an R-map.

Proof: Necessity

Let A be a regular closed set of Y and f:X→Y be an almost πgr-continuous function. Then $f^{-1}(A)$ is πgr-closed in X. Since X is a πgr-$T_{1/2}$-space ,$f^{-1}(A)$ is regular closed in X. Hence f is an R-map.

Sufficiency:

Suppose that f is an R-map and let A be a regular closed subset of Y. Then $f^{-1}(A)$ is regular closed in X. Since every regular closed set is πgr-closed, then $f^{-1}(A)$ is πgr-closed. Therefore, f is almost πgr-continuous.

Result:4.17

Every almost πgr-continuous function is almost πgb-continuous, almost πgα-continuous, almost πg-continuous, almost π*g-continuous,almost gpr-continuous.

Remark:4.18

The converse of the above need not be true as seen in the following examples.

Example: 4.19

X = { a,b,c,d}=Y, τ = {φ, X, {a}, {c,d} , {a,c,d}},σ={φ, Y,{a,b},{c},{a,b,c}},
Let f:X→Y be an identity map .Here f is almost πgb-continuous but not almost πgr-continuous.

Example: 4.20

X = { a,b,c,d}=Y, τ = {φ, X, {a}, {d},{a,d},{a,c},{a,c,d}},σ={φ, Y,{b},{a,c,d}},
Let f:X→Y be an identity map .Here f is almost gpr-continuous but not almost πgr-continuous.

Example: 4.21

X = { a,b,c,d}=Y, τ ={ φ, X, {a}, {d},{a,d},{a,c},{a,c,d}},,σ={φ, Y,{a,b,d},{c}},
Let f:X→Y be an identity map .Here f is almost πgα-continuous but not almost πgr-continuous.

Example: 4.22

X = { a,b,c,d}=Y, τ = {φ, X, {a}, {c,d} , {a,c,d}},σ={φ, Y,{a,b,d},{c}},
Let f:X→Y be an identity map .Here f is almost π*g-continuous but not almost πgr-continuous.

Example: 4.23

X = { a,b,c,d}=Y, τ = {φ, X, {a},{b},{a,b},{a,d},{a,b,d}, {a,c,d}},σ={φ, Y,{a,c,d},{b}},
Let f:X→Y be an identity map .Here f is almost πg-continuous but not almost πgr-continuous.

Proposition:4.24

If f is πgr-irresolute, then it is almost- πgr-continuous.

Proof: Straight forward.

Remark :4.25

The converse of the above need not be true as seen in the following example.

Example:4.26

Let X =Y= { a,b,c,d}, τ = {φ, X, {a}, {d} , {a,d} , {a,c,d},{a,b,d}},
σ={ φ,Y,{a},{c,d},{a,c,d}}.Let f : (X, τ) →(Y, σ) be defined by f(a)=b,f(b)=a,f(c)=c,f(d)=d.
The function f is almost-πgr-continuous but not πgr-irresolute.

Definition:4.27

A function f : (X, τ) →(Y, σ) is called a π-open map[4]if f(U) is π-open in (Y,σ) for every π-open set U in (X, τ)

Proposition:4.28

If f is bijective, π-open , almost- πgr-continuous, then f is πgr-irresolute.

Proof:

Let F be a πgr-closed set of Y. Let $f^{-1}(F) \subset U$, where U is π-open in X. Then $F \subset f(U)$. Since f is π-open, f(U) is π-open in Y, F is πgr-closed set in Y and $F \subset f(U) \Rightarrow rcl(F) \subset f(U)$
(i.e) $f^{-1}(rcl(F)) \subset U$. Since f is almost- πgr-continuous, $rcl(f^{-1}(rcl(F))) \subset U$.
So, $rcl(f^{-1}(F)) \subset rcl\ (f^{-1}(rcl(F))) \subset U$.
$\Rightarrow f^{-1}(F)$ is πgr-closed in X. Hence f is πgr-irresolute.

Corollary:4.29

Let a bijection f:(X, τ) →(Y, σ) be π-open , almost πgr-continuous and pre-regular closed.If X is πgr-$T_{1/2}$ -space, then (Y,σ) is πgr-$T_{1/2}$-space.

Proof:

Let F be πgr-closed subset of Y. By proposition 4.28, $f^{-1}(F)$ is πgr-closed in X. Since X is πgr-$T_{1/2}$-space, $f^{-1}(F)$ is regular closed in X. Since f is bijective, pre-regular closed, $F=f(f^{-1}(F))$ is regular closed in Y. Hence Y is πgr-$T_{1/2}$-space.

Proposition:4.30

If f is bijective, π-open , R-map, then f is πgr-irresolute.

Proof:

Since f is an R-map ,it is almost πgr-continuous. By proposition 4.28, f is πgr-irresolute.

Corollary:4.31

Let a bijection f:(X, τ) →(Y, σ) be π-open , R-map and pre-regular closed.If X is πgr-$T_{1/2}$ -space, then (Y,σ) is πgr-$T_{1/2}$-space.

Proof: Obvious.

5.πgr-compactness.

Definition: 5.1

A collection $\{A_i: i\in A\}$ of πgr-open sets in a topological space X is called a πgr-open cover of a subset S if $S\subset\cup\{A_i/i\in A\}$ holds.

Definition : 5.2

A topological space (X, τ) is called πgr-compact if every πgr-open cover of X has a finite subcover.

Definition: 5.3

A subset S of a topological space X is said to be πgr-compact relative to X, if for every collection $\{A_i: i\in A\}$ of πgr-open subsets of X such that $S\subset\cup\{A_i/i\in A\}$, there exists a finite subset Λ_o of A such that $S\subset\cup\{A_i/i\in \Lambda_o\}$

Definition:5.4

A subset S of a topological space X is said to be πgr-compact if S is πgr -compact as a subspace of X.

Proposition: 5.5

A πgr-closed subset of πgr-compact space is πgr-compact relative to X.

Proof:

Let A be a πgr-closed subset of a πgr-compact space X. Then X–A is πgr-open.Let Θ be a πgr-open cover for A. Then { Θ, X–A} is a πgr-open cover for X. Since X is πgr-compact, it has a finite subcover, say$\{P_1,P_2,\ldots, P_n\} = \Theta_1$.
If X-A $\notin \Theta_1$, then Θ_1 is a finite subcover of A. If X–A $\in \Theta_1$, then Θ_1– (X–A) is a subcover of A. Thus A is πgr-compact to relative to X.

Proposition: 5.6

Let f: (X, τ) →(Y, σ) be a surjective, πgr-continuous map. If X is πgr-compact, then Y is compact.

Proof:

Let $\{A_i : i\in A\}$ be an open cover of Y. Then $\{f^{-1}(A_1): i\in A\}$ is a πgr-open cover of X. Since X is πgr-compact, it has a finite subcover, say $\{f^{-1}(A_1), f^{-1}(A_2),\ldots, f^{-1}(A_n)\}$. Surjectiveness of f implies $\{A_1, A_2,\ldots, A_n\}$ is an open cover of Y and hence Y is compact.

Proposition: 5.7

If a map f: (X, τ) →(Y, σ) is πgr-irresolute and a subset S of X is πgr-compact relative to X, then the image f(S) is πgr-compact relative to Y.

Proof:

Let $\{A_i:i\in A\}$ be a collection of πgr-open sets in Y such that $f(S)\subset\cup\{A_i/i\in A\}$. Then $S\subset\cup\{f^{-1}(A_i):i\in A\}$, where $f^{-1}(A_i)$ is πgr-open in X for each i. Since S is πgr-compact relative to X, there exists a finite sub collection $\{A_1, A_2,\ldots, A_n\}$ such that $S\subset\cup\{f^{-1}(A_i):i=1,2,\ldots, n\}$

That is $f(S) \subset \cup \{(A_i): i= 1,2,\ldots,n\}$.Hence f(S) is πgr-compact relative to Y.

Lemma :5.8

Let θ: X×Y →X be a projection . If A is πgr-closed in X, then $\theta^{-1}(A) = A\times Y$ is πgr-closed in X×Y.

Proof:

Suppose A×Y ⊂O, where O is π-open in X×Y. Then O= U×Y, where U is π-open in X .Since U is a π-open set in X containing A and A is πgr-closed in X, we have $rcl_x(A) \subset U$.The above implies $rcl_{X\times Y}(A\times Y) \subset U\times Y$

(i.e) $rcl_{X\times Y}(A\times Y) \subset U\times Y$.Hence $A\times Y = \theta^{-1}(A)$ is πgr-closed in X×Y.

Theorem:5.9

If the product space of two non-empty spaces is πgr-compact , then each factor space is πgr-compact.

Proof:

Let X×Y be the product space of the non-empty spaces X and Y and suppose X×Y is a πgr-compact. Then the projection θ:X×Y→X is a πgr-irresolute map.

Hence θ(X×Y) = X is πgr-compact.

Similarly , we prove for the space Y.

6.πgr-connectedness.

Definition:6.1

A topological space (X,τ) is said to be πgr-connected if X cannot be written as the disjoint union of two non-empty πgr-open sets.

A subset of X is πgr-connected if it is πgr-connected as a subspace.

Proposition:6. 2

For a topological space X, the following are equivalent.

(i) X is πgr- connected.
(ii) The only subsets of X which are both πgr-open and πgr-closed are the empty set φ and X.
(iii) Each πgr -continuous map of X into a discrete space Y with atleast two points is a constant map.

Proof:

(i) ⟹(ii): Suppose S⊂X is a proper subset which is both πgr-open and πgr-closed. Then its complement X–S is also πgr-open and πgr-closed. Then X=S∪(X–S), a disjoint union of two non-empty πgr-open sets which contradicts the fact that X is πgr-connected. Hence S=φ or X.

(ii) ⟹(i): Suppose X= A∪B, where A∩B= φ, A ≠ φ,B ≠ φ and A and B are πgr-open. Since A= X–B, A is πgr-closed but by assumption A = φ or X, which is a contradiction. Hence (i) holds.

(ii)⟹(iii):Let f: X→Y be a πgr-continuous map, where Y is a discrete space with atleast two points. Then $f^{-1}(y)$ is πgr-closed and πgr-open for each y∈Y and $X=\cup\{f^{-1}(y): y\in Y\}$. By assumption , $f^{-1}(y) = \varphi$ for all y∈Y, then f will not be a map.Also, there cannot exist more than one y∈Y such that $f^{-1}(y) = X$. Hence , there exists only one y∈Y such that $f^{-1}(y) = X$ and $f^{-1}(y_1) = \varphi$, where $y\neq y_1 \in Y$. This shows that f is a constant map.

(iii) ⟹(ii) :Let S be both πgr-open and πgr-closed set in X. Suppose S≠φ. Let f: X→Y be a πgr-continuous map defined by f(S) = {a}, f(X-S) = {b}, where a≠b and a,b ∈Y. By assumption, f is constant. Therefore, S=X.

Remark:6.3

Every πgr-connected space is regular connected but the converse is not true as seen in the following example.

Example:6.4

Let X = { a,b,c}, τ = {φ, X, {a,b}, {a}}. Here the space X is regular connected.

The space X is not πgr-connected, since every subset of X is both πgr-open and πgr-closed.

Remark:6.5

πgr-connectedness and connectedness are independent.

Example:6.6

Let X={ a,b,c,d}, τ = {φ, X, {a},{b}, {a,b},{a,d},{a,b,d},{a,c,d}}. Here the space is not connected, since {a,c,d} ,{b} are both open and closed. But no subset of X is both πgr-closed and πgr-open.Hence the space X is πgr-connected.

Example:6.7

Let X= {a,b,c}, τ = {φ, X, {a,b}}. Here the space is connected. But every subset of X is both πgr-closed and πgr -open.Hence the space X is not πgr-connected

Proposition:6.8

If X is topological space with atleast two points and if π-open (X)= π-closed (X), then X is not πgr-connected.

Proof:

Since π-open (X) = π-closed (X), then there exists a proper subset of X, which is both πgr-open and πgr-closed. Hence the space X is not πgr-connected.

Proposition:6.9

Suppose X is a topological space with $\tau^{*}_{\pi gr} = \tau$, then X is regular connected iff X is πgr-connected .

Proof:

Follows from the definitions.

Proposition: 6.10

(i) If f:X→Y is πgr-continuous and onto, X is πgr-connected, then Y is regular connected.

(ii) If f : X→Y is πgr-irresolute and onto , X is πgr-connected, then Y is πgr-connected.

Proof:

Assume the contrary. Suppose Y is not regular connected. Then Y=A ∪ B, where A∩B =φ, A≠ φ , B≠ φ and A and B are regular open in Y. Since f is πgr-continuous and onto,
$X=f^{-1}(A) \cup f^{-1}(B)$ are disjoint non-empty πgr-open subsets of X. This contradicts the fact that X is πgr-connected. Hence the result.

(ii)Obvious

References

[1] Aslim.G.Caksu Guler.A and Noiri.T, On πgs-closed sets in topological spaces, Acta Math. Hungar., 112(4) (2006),275-283.
[2] Balachandran.K , P.Sundram and J.Maki, On generalized continuous maps in topological spaces, mem. Fac/. Sci. Kochi Univ. (math) 12(1991) , 5-13.
[3] Carnahan.D.A, Some properties related to compactness in topological spaces, Ph.D, Univ. of Arkansas.
[4] Dontchev.J, Noiri.T, Quasi normal spaces and πg-closed sets, Acta Math. Hungar 89(3), 2000, 211-219.
[5] Gnanambal.Y, On gpr-continuous functions in topological spaces, Indian J.Pure and appl.Math., 30(6):581-593,June 1999.
[6] Ganes M Pandya,Janaki.C and Arokiarani.I, On π*g-closed sets in topological spaces, Antarctica J.Math.,7(^)(2010),597-604.
[7] Hussain.T, Almost Continuous mappings, Prace Mat.,10(1966),1-7.
[8] Janaki.C, Studies on πgα-closed sets in topology, Ph.D Thesis,Bharathiar University, Coimbatore(2009)
[9] Jeyanthi.V and Janaki.C, πgr-closed sets in topological spaces, Asian Journal of Current Engg. Maths 1:5 sep 2012, 241-246.
[10] Levine.N,generalized closed sets in topology,Rend.circ. Mat .Palermo, 19(1970),89-96.
[11] Palaniappan.N and Rao.K.C, On Regular closed sets in topological spaces, Kyungpook Math .J. 33(1993),211-219.
[12] Palaniappan.N, Regular generalized star open and generalized star regular open sets in topology, J.Indian Acad.Math. Vol.23, No.2(2001),197-204.
[13] Park.J.H , On πgp-closed sets in topological spaces, Acta Mathematica Hungarica, 12, (4),257-283.
[14] M.K.Singal and A.R. Singal, Yokahama Math. J.16(1968),63-73.
[15] Sreeja.D and Janaki.C,On πgb-closed sets in topological spaces ,Inter. J.Math.Archive- 2(8),2011,page 1314-1320.
[16] Zaitsev, On certain classes of topological spaces and their bicompactifications, Dokl.Akad.Nauk.SSSR., 178(1968),778-779.