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On The Existence Of Periodic Solutions Of Certain Fourth Order Differential Equations With Dealy

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Abstract

We derive existence results for the periodic boundary value problem

$$x^{(iv)} + a\dddot{x} + f(\dot{x})\ddot{x} + c\dot{x} + g(t, x(x-\tau)) = p(t)$$

$x(0) = x(2\pi)$, $\dot{x}(0) = \dot{x}(2\pi)$, $\ddot{x}(0) = \ddot{x}(2\pi)$, $\dddot{x}(0) = \dddot{x}(2\pi)$

using degree theoretic methods. The uniqueness of periodic solutions is also examined.

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1. Introduction

In this paper we study the periodic boundary value problem

$$x^{(iv)} + a\dddot{x} + f(\dot{x})\ddot{x} + c\dot{x} + g(t, x(t-\tau)] = p(t) \quad (1.1)$$

$x(0) = x(2\pi)$, $\dot{x}(0) = \dot{x}(2\pi)$, $\ddot{x}(0) = \ddot{x}(2\pi)$, $\dddot{x}(0) = \dddot{x}(2\pi)$

with fixed delay $\tau \in [0, 2\pi)$ Where $c \neq 0$ is a constant, p: $[0, 2\pi] \to R$ and g: $[0, 2\pi] \times \Re \to \Re$ are 2π periodic in t and g satisfies certain Caratheodory conditions.

The unknown function x: $[0, 2\pi] \to R$ is defined for $0 < t < \tau$ by $x(t-\tau) = x(2\pi - (t-\tau))$

The differential equation $x^{(iv)} + a\dddot{x} + b\ddot{x} + h(x)\dot{x} + g(t, x(t-\tau)) = p(t)$ (1.2)

In which $b < 0$ is a constant was the object of a recent study [6].

Results on the existence and uniqueness of 2π periodic solutions were established subject to certain resonant conditions on g. Fourth order differential equations with delay occur in a variety of physical problems in fields such as Biology, Physics, Engineering and Medicine. In recent year, there have been many publications involving differential equation with delay; see for example [1,2,4,5,6,8,9]. However, as far as we know, there are few results on the existence and uniqueness of periodic solution to [1.1].

In what follows we shall use the spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ ' and $L^k([0, 2\pi])$ of continuous, k times continuously differentiable or measurable real functions whose kth power of the absolute value is Labesgue integrable.

We shall also make use of the sobolev space defined by

$H^k_{2\pi} = \{\{x : [0, 2\pi] \to R \mid x, \dot{x}$ are absolutely continuous on $[0, 2\pi]$ and, $\ddot{x} \in L^2[0, 2\pi]$ with norm $|x|^2_{H^2_{2\pi}} =$

$$\left(\frac{1}{2\pi}\int_o^{2\pi} x^2(t)dt\right)^2 + \frac{1}{2\pi}\sum_{i=1}^{2}\int_o^{2\pi} |x^i(t)|^2 dt.$$

$$x^i = \frac{d^i x}{dt^i}$$

2. The Linear cases

In this section we shall first consider the equation:

$$x^{(iv)}(t) + a\dddot{x}(t) + b\ddot{x}(t) + c\dot{x}(t) + dx(t-\tau) = 0 \quad (2.1)$$

$x(0) = x(2\pi)$, $\dot{x}(0) = \dot{x}(2\pi)$, $\ddot{x}(0) = \ddot{x}(2\pi)$, $\dddot{x}(0) = \dddot{x}(2\pi)$

Where a, b, c, d, are constants

Lemma 2.1 Let $c \neq 0$ and Let $a/c < 0$ Suppose that:

$$0 < d/c < n, \; n \geq 1 \quad (2.2)$$

Then (2.1) has no non-trivial 2π periodic solution for any fixed $\tau \in [0, 2\pi)$.

Proof

By substituting $x(t) = e^{\lambda t}$ with $\lambda = in$, $_i^2 = -1$. We can see that the conclusion of the Lemma is true if $\Phi(n, \tau) = an^3 - cn + d \operatorname{Sin} n\tau \neq 0$ for all $n \geq 1$ and $\tau \in [0, 2\pi)$ (2.3)

By (2.2) we have

$$c^{-1}\phi(n, \tau) = \frac{a}{c}n^3 - n + \frac{d}{c}\operatorname{Sin} n\tau \leq \frac{a}{c}n^3 - n + \frac{d}{c} \leq \frac{a}{c}n^3 < 0$$

Therefore $\Phi(n, \tau) \neq 0$ and the result follows

If $x \in L^1$ [0, 2π] we shall write

$$\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt, \tilde{x}(t) = x(t) - \bar{x}$$

So that

$$\int_0^{2\pi} \tilde{x}(t)dt = 0$$

We shall consider next the delay equation

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau) = 0 \qquad (2.4)$$

$x(0) = x(2\pi)$, $\dot{x}(0) =$, $\dot{x}(2\pi)$, $\ddot{x}(0) = \ddot{x}(2\pi)$, $\dddot{x}(0) = \dddot{x}(2\pi)$

Where a , b, c are constants and $d \in L^1_{2\pi}$

Here the coefficient d in (2.4) is not necessarily constant. We have he following results which apart from being of independent interest are also useful in the non-linear case involving (1.1)

Lemma 2.2 Let $c \neq 0$ and let $a/c < 0$ Set $\Gamma(t) = c^{-1}d(t) \in L^2_{2\pi}$ Suppose that

$$0 < \Gamma(t) < 1 \qquad (2.5)$$

Then for arbitrary constant b the equation (2.4) admits in $H^1_{2\pi}$ only the trivial solution for every $\tau \in [0, 2\pi)$.

We note that a and c are not arbitrary.

Proof

If $x \in H^1_{2\pi}$ is a possible solution of (2.4) then on multiplying (2.4) by $\bar{x} + \dot{\tilde{x}}$ (t) and integrating over [0, 2π] noting that

$$\frac{1}{2\pi}\int_0^{2\pi}(\bar{x} + \dot{\tilde{x}}(t))$$

$$c^{-1}\left[x^{(iv)} + a\dddot{x} + b\ddot{x}\right] = -\tfrac{1}{2\pi}\tfrac{a}{c}\int_o^{2\pi} \ddot{x}^2(t)dt$$

We have that

$$0 = \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} + \dot{\tilde{x}}(t)\left\{c^{-1}[x^{(iv)} + a\dddot{x} + b\ddot{x}\ ] + \dot{x} + \Gamma(t)x(t-\tau)\ \right\}dt$$

$$= -\frac{1}{2\pi}\frac{a}{c}\int_0^{2\pi}\ddot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi}(\bar{x} + \dot{\tilde{x}}(t))[\dot{x} + \Gamma(t)x(t-\tau)dt]$$

$$\geq \tfrac{1}{2\pi}\int_0^{2\pi}(\bar{x} + \dot{\tilde{x}}(t))\{\dot{x}(t) + \Gamma(t)x(t-\tau)\}dt$$

$$= \frac{1}{2\pi}\int_o^{2\pi}\dot{\tilde{x}}^2(t)dt + \int_0^{2\pi}\Gamma(t)\dot{\tilde{x}}x(t-\tau)dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}^2dt$$

$$+ \frac{1}{2\pi}\int_o^{2\pi}\Gamma(t)\bar{x}\tilde{x}(t-\tau)dt$$

Using the identity

$$ab = \frac{(a+b)^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

We get

$$= \frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}^2dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}\tilde{x}(t-\tau)dt$$

$$+ \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\left\{\frac{[x(t-\tau) + \dot{\tilde{x}}(t)]^2}{2} - \frac{\dot{\tilde{x}}^2}{2} - \frac{\tilde{x}^2(t-\tau)}{2} - \bar{x}\tilde{x}(t-\tau) - \frac{\bar{x}^2}{2}\right\}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt - \frac{1}{2\pi}\int_0^{2\pi}\frac{\Gamma(t)}{2}\left[\dot{\tilde{x}}^2 + \tilde{x}^2(t-\tau)\right]dt +$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\Gamma(t)}{2}\ \left\{\ \left[x(t-\tau) + \dot{\tilde{x}}(t)\right]^2 + \bar{x}^2\ \right\}dt$$

Using (2.5) we get

$$0 \geq \tfrac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt - \tfrac{1}{2\pi}\int_0^{2\pi}\frac{\Gamma(t)}{2}\left[\dot{\tilde{x}}^2 + \tilde{x}^2(t-\tau)\right]dt$$

From the periodicity of $\dot{\tilde{x}}$ we have that

$$\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt = \int_0^{2\pi}\dot{\tilde{x}}^2(t-\tau)dt$$

Hence

$$0 \geq \tfrac{1}{2}[\tfrac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t-\tau) - \Gamma(t)\tilde{x}^2(t-\tau)]dt +$$

$$\tfrac{1}{2}\ [\tfrac{1}{2\pi}\int_o^{2\pi}(\dot{\tilde{x}}^2(t) - \Gamma(t)\dot{\tilde{x}}^2(t-\tau)]dt \qquad (2.6)$$

Using (2.5) we can see that the last expression is non-negative hence

$$0 \geq \tfrac{1}{2}[\tfrac{1}{2\pi}\int_0^{2\pi}(\dot{\tilde{x}}^2(t-\tau) - \Gamma(t)\tilde{x}^2(t-\tau)]dt$$

$$\geq \delta|\tilde{x}|^2_{H^1_{2\pi}}$$

By Lemma 1 of [8] where $\delta > 0$ is a constant. This implies $\tilde{x} = 0$ a.e and that x = $\bar{x}$.
But a constant map cannot be a solution of (2.4) since $\Gamma(t) \neq 0$
Thus x = 0

Theorem 2.1
Let all the conditions of Lemma 2.2 hold and let δ be related to Γ by Lemma 2.2.Suppose further that $V \in L^2_{2\pi}$ satisfies $0 \leq V(t) \leq \Gamma(t) + \varepsilon$ a.e t $\in [0,2\pi]$ where $\varepsilon >$ 0 then

$$\frac{1}{2\pi}\int_0^{2\pi}(\bar{x}+\dot{\tilde{x}}(t))\{c^{-1}[x^{(iv)} + a\ddot{x} + f(\dot{x})\ddot{x}] + \dot{x} + V(t)x(t-\tau)\ \}dt$$

$$\geq (\delta - \varepsilon)|\tilde{x}\ |^2_{H^1_{2\pi}} \qquad (2.7)$$

Proof

We have from the proof of Lemma 2.2 that

$$\frac{1}{2\pi}\int_0^{2\pi}(\bar{x}+\dot{\tilde{x}}(t))\{c^{-1}[x^{(iv)} + a\ddot{x} + f(\dot{x})\ddot{x}] + \dot{x} + V(t)x(t-\tau)\ \}dt$$

$$\geq \frac{1}{2}(\frac{1}{2\pi}\int_0^{2\pi}(\dot{\tilde{x}}^2(t-\tau) - V(t)\tilde{x}^2(t-\tau)\ )dt + \frac{1}{2}(\frac{1}{2\pi}\int_0^{2\pi}(\dot{\tilde{x}}^2(t) - V(t)\tilde{x}^2(t))dt$$

$$\geq \frac{1}{2}(\tfrac{1}{2\pi}\int_0^{2\pi}(\dot{\tilde{x}}^2(t-\tau) - \Gamma(t)\tilde{x}^2(t-\tau))dt - \tfrac{\varepsilon}{2}(\tfrac{1}{2\pi}\int_0^{2\pi}\tilde{x}^2(t-\tau))dt$$

$$+\frac{1}{2}(\tfrac{1}{2\pi}\int_0^{2\pi}(\dot{\tilde{x}}^2(t) - \Gamma(t)\dot{\tilde{x}}^2(t))dt - \tfrac{\varepsilon}{2}(\tfrac{1}{2\pi}\int_0^{2\pi}\tilde{x}^2(t)dt$$

From condition (2.5) , Lemman 2.2 and Wirtinger's inequality we have

$$\geq \delta|\tilde{x}|_{H^1_{2\pi}} - \varepsilon|\tilde{x}|_{L^2_{2\pi}} \geq \delta|\tilde{x}|_{H^1_{2\pi}} - \varepsilon|\tilde{x}|_{H^1_{2\pi}} = (\delta - \varepsilon)|\tilde{x}|_{H^1_{2\pi}}$$

3. The Non Linear Case

We shall consider the non-linear delay equation

$$x^{(iv)} + a\ddot{x} + f(\dot{x})\ddot{x} + c\dot{x} + g(t, x(t-\tau)) = p(t) \qquad (3.1)$$

$$x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi), \ddot{x}(0) = \ddot{x}(2\pi), \dddot{x}(0) = \dddot{x}(2\pi)$$

where $f : \Re \to \Re$ is a continuous function and $g : [0,2\pi]x\Re \to \Re$ is such that g(. x) is a measurable on $[0,2\pi]$ for each x$\in \Re$ and $g(t,.)$ is continuous on $\Re$ for almost each t$\in$[0,2π]
We assume moreover that for each r > 0 there exists $Y_r \in L^1_{2\pi}$ such that

$$|g(t,x)| \leq Y_r(t) \qquad (3.2)$$

for a.e t$\in$ $[0,2\pi]$ and all x$\in [-r,r]$ such a g is said to satisfy the Caratheodory's condition.

Theorem 3.1
Let $c \neq 0$ and let a/c < 0
Suppose that g is a caratheodory function satisfying the inequalities

$$c^{-1}xg(t,x) \geq 0 \qquad (|x| \geq r) \qquad (3.3)$$

$$\text{Lim}\ \sup_{|x|\to\infty} \frac{g(t,x)}{cx} \leq \Gamma(t) \qquad (3.4)$$

Uniformly a.e t $\in [0,2\pi]$ where r > 0 is constant and $\Gamma \in L^2_{2\pi}$ is such that $0 < \Gamma(t) < 1$.
Suppose $p \in L^2_{2\pi}$ is such that $\bar{p} = \frac{1}{2\pi}\int_0^{2\pi} p(t)dt =$ 0 then for arbitrary continuous function f the boundary value problem (3.1) has at least one solution for every $\tau \in [0,2\pi)$

Proof
Let $\delta > 0$ be related to Γ as in Lemma 2.2 so that by (3.3) and (3.4) there exists a constant R_1 such that for IxI $\geq R_1$

$$0 \leq \frac{g(t,x)}{cx} \leq \Gamma(t) + \delta/2 \qquad (3.5)$$

Define $\tilde{y}(t,x)$ by

$$\tilde{y}(t,x)=\begin{cases}(cx)^{-1}g(t,x) & |x|\geq R_1\\ (cR_1)^{-1}g(t,R_1) & 0<x<R_1\\ -(cR_1)^{-1}g(t,-R_1) & -R_1<x<0\\ \Gamma(t) & x=0\end{cases} \tag{3.6}$$

Then

$$0<\tilde{y}(t,x)\leq\Gamma(t)+\delta/2 \tag{3.7}$$

for a .e t $\in[0,2\pi]$ and all x $\in\Re$. Moreover the function $\tilde{y}(t,x)$ satisfy Caratheodory's conditions and $\tilde{g}$: $[0,2\pi]\times\Re\to\Re$ defined by

$$\tilde{g}(t,x(t-\tau))=g(t,x(t-\tau))-cx(t-\tau)\tilde{y}(t,x(t-\tau)) \tag{3.8}$$

is such that for a.e t$\in[0,2\pi]$ and all x$\in\Re$. $|\tilde{g}(t,x(t-\tau))|\leq\alpha(t)$ for some $\alpha(t)\in L^2_{2\pi}$.

Let $\lambda\in[0,1]$ be such that

$$c^{-1}[x^{(iv)}+a\dddot{x}+\lambda f(\dot{x})\ddot{x}]+\dot{x}+(1-\lambda)\Gamma(t)x(t-\tau)+\lambda\tilde{y}(t,x(t-\tau))x(t-\tau)+c^{-1}(1-\lambda)b\ddot{x}+\lambda c^{-1}\tilde{g}(t,x(t-\tau))-c^{-1}\lambda p(t)=0 \tag{3.9}$$

For $\lambda=0$ we obtain (2.1) which by Lemma 2.2 admits only the trivial solution

For $\lambda=1$ we get the original equation (1.1). To prove that equation (3.1) has at least one solution, we show according to the Leray-Shauder Method that the possible solution of the family of equations (3.9) are apriori bounded in $C^3[0,2\pi]$ independently of $\lambda\in[0,1]$.

Notice that by (3.5) one has

$$0\leq(1-\lambda)\Gamma(t)+\lambda\tilde{y}(t,x(t-\tau))\leq\Gamma(t)+\delta/2 \tag{3.10}$$

Then using Theorem 2.1 with $V(t)=(1-\lambda)\Gamma(t)+\lambda\tilde{y}(t,x(t-\tau))$ and Cauchy Schwarz inequality we get

$$0=\frac{1}{2\pi}\int_0^{2\pi}(\bar{x}+\tilde{x}(t))\{c^{-1}[x^{(iv)}+a\dddot{x}+\lambda f(\dot{x})\ddot{x}]+\dot{x}+(1-\lambda)\Gamma(t)x(t-\tau)+\lambda\tilde{y}(t,x(t-\tau))+\lambda\tilde{g}(t,x(t-\tau))+(1-\lambda)b\ddot{x}-\lambda p(t)\}dt.$$

$$\geq\delta/2|\tilde{x}|^2_{H^1_{2\pi}}-(|\alpha|_2+|p|_2)(|\bar{x}|+|\tilde{x}|_2)$$

$$\geq\delta/2|\tilde{x}|^2_{H^1_{2\pi}}-\beta(|\bar{x}|+|\tilde{x}|_{H^1_{2\pi}})$$

Thus

$$|\tilde{x}|^2_{H^1_{2\pi}}\leq\frac{2\beta}{\delta}(|\bar{x}|+|\tilde{x}|^2_{H^1_{2\pi}}) \tag{3.11}$$

With $\beta>0$ independent of x and λ. Integrating (3.9) over $[0,2\pi\}$We obtain

$$(1-\lambda)\int_0^{2\pi}\Gamma(t)x(t-\tau)dt=-c^{-1}\lambda\int_0^{2\pi}g(t,x(t-\tau)) \tag{3.12}$$

Since $\Gamma(t)>0$ we derive that

$$\frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)dt=\bar{\Gamma}>0 \tag{3.13}$$

Hence if $x(t)\geq r$ for all $t\in[0,2\pi]$, (3.3) and (3.12) implies that $(1-\lambda)\bar{\Gamma}<0$ contradicting $\bar{\Gamma}>0$. Similarly if $x(t)\leq -r$ for all $t\in[0,2\pi]$ we reach a contradiction.

Thus there exists a, t_1, $\in[0,2\pi]$ such that $|x(t_1)|<r$. Le t_2 be such that

$$\bar{x}=x(t_2)=x(t_1)+\int_{t_1}^{t_2}\dot{x}(s)ds.$$

This implies that $|\bar{x}|\leq r+2\pi|\tilde{x}|_{H^1_{2\pi}}$

Substituting this in (3.11) we get

$$|\tilde{x}|^2_{H^1_{2\pi}}\leq c_1|\tilde{x}|_{H^1_{2\pi}}$$

$$\text{or } |\tilde{x}|_{H^1_{2\pi}}\leq c_1, c_1>0 \tag{3.14}$$

Now

$$|x|_{H^1_{2\pi}}\leq|\bar{x}|+|\tilde{x}|_{H^1_{2\pi}}\leq r+(2\pi+1)c_1=c_2 \tag{3.15}$$

Thus

$$|\dot{x}|_2\leq c_3, c_3>0 \tag{3.16}$$

From (3.16)we have

$$|x|_\infty\leq c_4, c_4>0 \tag{3.17}$$

Multiplying (3.9) by $-\dot{x}(t)$ and integrating over $[0,2\pi]$ we have

$$|\ddot{x}|^2_2\leq|a|^{-1}(|\dot{x}|^2_2+|1+\tfrac{\delta}{2}||\dot{x}|_2|x|_\infty+|\alpha|_2|\dot{x}|_2+|p|_2|\dot{x}|_2)$$

Hence

$$|\ddot{x}|_2\leq c_5, c_5>0 \tag{3.18}$$

And thus

$$|\dot{x}|_\infty\leq c_6, c_6>0 \tag{3.19}$$

Multiplying (3.9) by $-\dddot{x}(t)$ and integrating over $[0,2\pi]$
We get

$$|\ddot{x}|_2^2 \le |f(\ddot{x})|_\infty |\ddot{x}|_2^2 + |1+\delta/2||\ddot{x}|_2|x|_\infty + |c|^{-1}|\alpha|_2|\ddot{x}|_2 + |p|_2|\ddot{x}|_2 + |b||\ddot{x}|_2$$

Thus

$$|\ddot{x}|_2 \le c_7, c_7 > 0 \qquad (3.20)$$

And hence

$$|\ddot{x}|_\infty \le c_8, c_8 > 0 \qquad (3.21)$$

Also

$$|x^{(iv)}|_I \le c_9, c_9 > 0 \qquad (3.22)$$

Since $\ddot{x}(0) = \ddot{x}(2\pi)$ there exists $t_o \in [0,2\pi]$ Such that $\dddot{x}(t_o) = 0$ Hence

$$|\dddot{x}|_\infty \le c_{10}, c_{10} > 0 \qquad (3.23)$$

From (3.17), (3.19), (3.22) and (3.23) our result follows.

4. Uniqueness Result

If in (1.1), $f(\dot{x}) = b$ a constant. The following uniqueness results holds.

Theorem 4.1
Let a, b, c, be constants with $c \neq 0$ $a/c < 0$.
Suppose g is a caratheodony function satisfying

$$0 < \frac{g(t, x_1(t-\tau)) - g(t, x_2(t-\tau))}{c(x_1 - x_2)} < \Gamma(t)$$

For a.e., $t \in [0,2\pi]$ and all $x_1, x_2 \in R$ $x_1 \neq x_2$
where $\Gamma \in L^2_{2\pi}$
Then the boundary value problem

$$x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + g(t, x(t-\tau)) = p(t)$$
$$x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi), \ddot{x}(0) = \ddot{x}(2\pi), \dddot{x}(0) = \dddot{x}(2\pi)$$
(4.1)

has art most one solution.

Proof
Let $u = x_1 - x_2$ for any two solutions x_1, x_2 of (4.1).
Then u satisfies the boundary value problem

$$c^{-1}[u^{(iv)} + a\dddot{u} + b\ddot{u}] + \dot{u} + \beta(t)u(t-\tau) = 0$$
$$u(0) = u(2\pi), \dot{u}(0) = \dot{u}(2\pi), \ddot{u}(0) = \ddot{u}(2\pi), \dddot{u}(0) = \dddot{u}(2\pi)$$

Where $\beta(t) \in L^2_{2\pi}$ is defined by

$$\beta(t) = \frac{g(t, x_1(t-\tau)) - g(t, x_2(t-\tau))}{c(x_1 - x_2)}$$

If $u = x_1 - x_2 \neq 0$ and since $0 < \beta(t) \le \Gamma(t)$ for a.e $t \in [0,2\pi]$ then using the arguments of theorem 2.1 we have that u = 0 and thus $x_1 = x_2$ a. e.

REFERENCES

[1] **F. Ahmad.** Linear delay differential equations with a positive and negative term. *Electronic Journal of Differential Equations. Vol. 2003 (2003) No. 9, 1-6.*

[2] **J.G. Dix.** Asymptotic behaviour of solutions to a first order differential equations with variable delays. *Computer and Mathematics with applications Vol 50 (2005) 1791 - 1800*

[3] **R. Gaines and J. Mawhin,** Coincidence degree and Non-linear differential equations, *Lecture Notes in Math, No.568 Springer Berlin, (1977).*

[4] **S.A. Iyase:** On the existence of periodic solutions of certain third order Non-linear differential equation with delay. *Journal of the Nigerian Mathematical Society Vol. 11, No. 1 (1992) 27 - 35*

[5] **S.A. Iyase.** Non-resonant oscillations for some fourth-order differential equations with delay. *Mathematical Proceedings of theRoyal Irish Academy***,** Vol.99A, No.1, 1999, 113-121

[6] **S.A. Iyase and P.O.K. Aiyelo,** Resonant oscillation of certain fourth order Nonlinear differential equations with delay, *International Journal of Mathematics and Computation Vol.3 No. J09, June 2009 p67-75*

[7] **Oguztoreli and Stein,** An analysis of oscillation in neuromuscular systems. *Journal of Mathematical Biology 2 1975 87-105.*

[8] **E.De. Pascal and R. Iannaci:** Periodic solutions of generalized Lienard equations with delay, *Proceedings equadiff 82, Wurzburg (1982) 148 - 156*

[9] **H.O. Tejumola,** Existence of periodic solutions of certain third order non-linear differential equations with delay. *Journal of Nigerian Mathematical Society Vol. 7 (1988) 59-66*