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Iterative Method For The Solution Of Simple And Multiple Roots Of Non Linear System Of Equations

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Abstract

In this paper a method called Parametric Method of Iteration is developed for solving the non linear system of equations and showed the efficiency of this method over iteration method and Newton Raphson method by considering some examples.

Key words*:* N-R Method, Iterative Method

I. INTRODUCTION

It is well known that the solution of systems of non-linear equations play an important role in all the fields of science and engineering applications and there exist a great variety of problems for the solution of such problems. Many applications in Science and Engineering are described by a set of n-coupled, non-linear algebraic equations in n-variables $x_1, x_2, ..., x_n$ of the form

$$F_1(x_1, x_2, ..., x_n) = 0$$
$$F_2(x_1, x_2, ..., x_n) = 0 \qquad (1.1)$$
$$F_n(x_1, x_2, ..., x_n) = 0$$

The equations (1.1) can be expressed as a system

$$F_i(X) = 0$$
$$(i = 1, 2, ..., n)$$

(1.2) where X is an n-dimensional vector.

Though we have various techniques like the method of steepest discent, Riks/Wempner method, Von-Mises method, Power method, Gradient method and Graphical method [1,2] for the solution of (1.2), the most generally used methods for its solution are the successive approximation method also known as method of iteration, and the multivariable Newton-Raphson method.

In the Successive approximation mehod the system (1.2) is written as

$$x_i = f_i(x_1, x_2, ..., x_n) \qquad (1.3)$$
$$(i = 1, 2, ..., n)$$

If $x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}$ be the initial approximation to the solution of (1.2), then in this iterative method the improved values are found as indicated below

$$x_1^{(k+1)} = f_1(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, ..., x_n^{(k)})$$

$$x_2^{(k+1)} = f_2(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, ..., x_n^{(k)})$$
$$x_3^{(k+1)} = f_3(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, ..., x_n^{(k)})$$
$$x_n^{(k+1)} = f_n(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, ..., x_n^{(k)}), (k=0,1,2,\ldots) \qquad (1.4)$$

As is well known that the iterations (1.4) are to be repeated successively until the convergence is achieved upto the desired accuracy and this method convergences under the condition.

$$\sum_{j=1}^{n} \left| \frac{\partial f_i(X)}{\partial x_j} \right|_{x=x^{(k)}} < 1 \qquad (1.5)$$

(i=1,2,….,n), for each k.

whereas in the multivariable Newton-Raphson method, the system (1.2) is solved by iterating the scheme

$$x^{(k+1)} = x^{(k)} - \left[J^{(k)} \right]^{-1} F(x^{(k)}) \qquad (1.6)$$

(k=0,1,2,..)

where the Jacobian $J^{(k)}$ is

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & . & . & . & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & . & . & . & \frac{\partial F_2}{\partial x_n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & . & . & . & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

The multivariable Newton-Raphson method converges if the functions $F_i(X)$, all have continuous first order partial derivatives near a root X^*, and if the Jacobian is non-singular in the

neighbourhood of this root and if $X^{(k)}$ is taken sufficiently close to the solution X^*.
In this paper we develop a method for the solution of the simultaneous non-linear equations (1.1) which will be presented in the next section. This method is also compared with the other methods by considering some numerical examples.

2. PARAMETRIC METHOD OF ITERATION

To solve the system (1.1), we rewrite each equation of (1.1) by introducing a set of parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ all of them are positive, in the form

$$\begin{aligned} x_1 &= (1-\alpha_1)x_1 + \alpha_1 f_1(x_1, x_2, \ldots, x_n) \\ x_2 &= (1-\alpha_2)x_2 + \alpha_2 f_2(x_1, x_2, \ldots, x_n) \qquad (2.1) \\ x_n &= (1-\alpha_n)x_n + \alpha_n f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

For the solution of the system (1.1), the method is defined as

$$\begin{aligned} x_1^{(k+1)} &= (1-\alpha_1^{(k)})x_1^{(k)} + \alpha_1^{(k)} f_1(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}) \\ x_2^{(k+1)} &= (1-\alpha_2^{(k)})x_2^{(k)} + \alpha_2^{(k)} f_2(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}) \\ x_3^{(k+1)} &= (1-\alpha_3^{(k)})x_3^{(k)} + \alpha_3^{(k)} f_3(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}) \qquad (2.2) \\ x_n^{(k+1)} &= (1-\alpha_n^{(k)})x_n^{(k)} + \alpha_n^{(k)} f_n(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}) \end{aligned}$$

(k = 0,1,2,…)

The equations (2.2) are expressed as

$$\begin{aligned} x_1^{(k+1)} &= (1-\alpha_1^{(k)})x_1^{(k)} + \alpha_1^{(k)} f_1^{(k)} \\ x_2^{(k+1)} &= (1-\alpha_2^{(k)})x_2^{(k)} + \alpha_2^{(k)} f_2^{(k)} \\ x_3^{(k+1)} &= (1-\alpha_3^{(k)})x_3^{(k)} + \alpha_3^{(k)} f_3^{(k)} \qquad (2.3) \\ x_n^{(k+1)} &= (1-\alpha_n^{(k)})x_n^{(k)} + \alpha_n^{(k)} f_n^{(k)} \end{aligned}$$

It can directly be seen that the method (2.3) coincides with (1.4) when $\alpha_i^{(k)} = 1$ for each i and k. As the parameters $\alpha_i^{(k)}$ in (2.3) accelerates or improves the convergence of (1.4), the method (2.3) may be called as parametric method of iteration.

As given in [3] and [1], one can easily show that the method (2.3) converges when the conditions

$$\left|(1-\alpha_1) + \alpha_1 \frac{\partial f_1}{\partial x_1}\right| + \left|\alpha_2 \frac{\partial f_2}{\partial x_1}\right| + \ldots + \left|\alpha_n \frac{\partial f_n}{\partial x_1}\right| < 1$$

$$\left|\alpha_1 \frac{\partial f_1}{\partial x_2}\right| + \left|(1-\alpha_2) + \alpha_2 \frac{\partial f_2}{\partial x_2}\right| + \ldots + \left|\alpha_n \frac{\partial f_n}{\partial x_2}\right| < 1 \qquad (2.4)$$

$$\left|\alpha_1 \frac{\partial f_1}{\partial x_1}\right| + \left|\alpha_2 \frac{\partial f_2}{\partial x_n}\right| + \ldots + \left\|(1-\alpha_n) + \alpha_n \frac{\partial f_n}{\partial x_n}\right\| < 1$$

hold true for all $x = x^{(k)}$ and $\alpha = \alpha^{(k)}$ for each k.

If we choose

$$\alpha_i^{(k)} = \left[1 - \frac{\partial f_i}{\partial x_i}\Big|_{x=x^{(k)}}\right]^{-1}, \text{ (i=1,2,…,n), for each k} \qquad (2.5)$$

Then the iteration process (2.3) takes the form

$$x_i^{(k+1)} = \left[f_1^{(k)} - x_1^{(k)} \frac{\partial f_1}{\partial x_1}\Big|_{x=x^{(k)}}\right] \Big/ \left[1 - \frac{\partial f_1}{\partial x_1}\Big|_{x=x^{(k)}}\right]$$

$$x_2^{(k+1)} = \left[f_2^{(k)} - x_2^{(k)} \frac{\partial f_2}{\partial x_2}\Big|_{x=x^{(k)}}\right] \Big/ \left[1 - \frac{\partial f_2}{\partial x_2}\Big|_{x=x^{(k)}}\right] \qquad (2.6)$$

.

$$x_n^{(k+1)} = \left[f_n^{(k)} - x_n^{(k)} \frac{\partial f_n}{\partial x_n}\Big|_{x=x^{(k)}}\right] \Big/ \left[1 - \frac{\partial f_n}{\partial x_n}\Big|_{x=x^{(k)}}\right]$$

For the choices of α_i given (2.5) , the conditions for convergence of the method (2.6) can be obtained from (2.4) as

$$\left|\alpha_2 \frac{\partial f_2}{\partial x_1}\right| + \left|\alpha_3 \frac{\partial f_3}{\partial x_1}\right| + \ldots + \left|\alpha_n \frac{\partial f_n}{\partial x_1}\right| < 1$$

$$\left|\alpha_1 \frac{\partial f_1}{\partial x_2}\right| + \left|\alpha_3 \frac{\partial f_3}{\partial x_2}\right| + \ldots + \left|\alpha_n \frac{\partial f_n}{\partial x_2}\right| < 1 \qquad (2.7)$$

$$\left|\alpha_1 \frac{\partial f_1}{\partial x_n}\right| + \left|\alpha_2 \frac{\partial f_2}{\partial x_n}\right| + \ldots + \left|\alpha_{n-1} \frac{\partial f_{n-1}}{\partial x_n}\right| < 1$$

For all $x = x^{(k)}$ and $\alpha = \alpha^{(k)}$ for each k.
For solving non-linear equations in single variable of the form

F(x)=0 (2.8)

The Parametric method of iteration can be defined from (2.3) as

$$x^{(k+1)} = (1-\alpha^{(k)})x^{(k)} + \alpha^{(k)} f(x^{(k)}) \qquad (2.9)$$

Where α is a parameter, whose choice using (2.5) obtained as

$$\alpha^{(k)} = \frac{1}{\left[1 - f'(x)\big|_{x=x^{(k)}}\right]} \text{ for each k} \qquad (2.10)$$

With this choice of (2.9) the method (2.8) reduce to

$$x^{(k+1)} = \left[f(x^{(k)}) - x^{(k)} f'(x)\big|_{x=x^{(k)}} \right] / \left[1 - f'(x)\big|_{x=x^{(k)}}\right] \quad 2.11)$$

As it is well known that the Newton-Raphson method for the solution of (2.8) is given by

$$x^{(k+1)} = x^{(k)} - \frac{F(x^{(k)})}{F'(x)\big|_{x=x^{(k)}}} \qquad (2.12)$$

It is interesting to note that F(x) of equation (2.8) is expressed in the form

$$F(x) = x - f(x) \qquad (2.13)$$

Then the Newton-Raphson iteration (2.12) exactly takes the form (2.11).

3. NUMERICAL EXAMPLES:

We consider the non-linear system of three equations in three unknowns:

$$F_1(x_1, x_2, x_3) = 3x_1 - \cos x_2 x_3 - 0.5 = 0$$

$$F_2(x_1, x_2, x_3) = x_1^2 - 625x_2^2 = 0$$

$$F_3(x_1, x_2, x_3) = e^{-x_1 x_2} + 20x_3 + 9 = 0 \qquad (3.1)$$

As noted by William W.Hager [4], this system has more than one solution. Expressing the system (3.1) as

$$x_1 = f_1(x_1, x_2, x_3)$$

$$x_2 = f_2(x_1, x_2, x_3) \qquad (3.2)$$

$$x_3 = f_3(x_1, x_2, x_3)$$

where

$$f_1 = (\cos x_2 x_3 + 0.5 + x_1)/4,$$

$$^* f_2 = \pm x_1 / 25,$$

$$f_3 = (x_3 - 9 - e^{-x_1 x_2)})/21$$

(*Here + or – sign should be taken in accordance with the initial guess)

The parametric method of iteration (2.6) for the solution of (3.1) can be written as

$$x_1^{(k+1)} = k\left[f_1^{(k)} - x_1^{(k)} \frac{\partial f_1}{\partial x_1}\Big| x = x_k \right] / \left[1 - \frac{\partial f_1}{\partial x_1}\Big| x = x_k \right]$$

$$x_2^{(k+1)} = k\left[f_2^{(k)} - x_2^{(k)} \frac{\partial f_2}{\partial x_2}\Big| x = x_k \right] / \left[1 - \frac{\partial f_2}{\partial x_2}\Big| x = x_k \right]$$

$$x_3^{(k+1)} = k\left[f_3^{(k)} - x_3^{(k)} \frac{\partial f_3}{\partial x_3}\Big| x = x_k \right] / \left[1 - \frac{\partial f_3}{\partial x_3}\Big| x = x_k \right] \qquad (3.3)$$

The iteration method for solving (3.1) can be taken by writing (3.1) into the form (3.2) as

$$x_1^{(k+1)} = (\cos x_2^{(k)} x_3^{(k)} + 0.5 + x_1^{(k)})/4,$$

$$^* x_2^{(k+1)} = \pm x_1^{(k+1)}/25, \qquad (3.4)$$

$$x_3^{(k+1)} = (x_3^{(k)} - 9 - e^{-x_1^{(k)} x_2^{(k)}})/21$$

(*Here + or – sign should be taken in accordance with the initial guess)

And, the Newton-Raphson iteration for the solution of (3.1) is

$$x_1^{(k+1)} = x^{(k)} - \left[J(x^{(k)}) \right]^{-1} F(x^{(k)}) \qquad (3.5)$$

Where $X = (x_1, x_2, x_3)^T$ and the Jacobian J is a 3x3 matrix:

$$J(x) = \begin{bmatrix} \frac{\partial F_1(X)}{\partial x_1} & \frac{\partial F_1(X)}{\partial x_2} & \frac{\partial F_1(X)}{\partial x_3} \\ \frac{\partial F_2(X)}{\partial x_1} & \frac{\partial F_2(X)}{\partial x_2} & \frac{\partial F_2(X)}{\partial x_3} \\ \frac{\partial F_3(X)}{\partial x_1} & \frac{\partial F_3(X)}{\partial x_2} & \frac{\partial F_3(X)}{\partial x_3} \end{bmatrix} \qquad (3.6)$$

to compare the methods (3.3), (3.4) and (3.5) for the solution of (3.1), we tabulate the following results for various initial approximation to the unknowns

TABLE 3.1
For $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$

Iteration Number	Variable	Iteration Method	Newton- Raphson Method	Parametric Method of Iteration
1	X_1 X_2 X_3	0.375 0.015 -0.475923371	Since J^{-1} does not exist, Newton-Raphson Method fails to converge	0.5 0.02 -0.4995024917
2	X_1 X_2 X_3	0.4687436296 0.01874974519 -0.4984368122	---	0.4999833666 0.01999933467 -0.4995025246
3	X_1 X_2 X_3	0.49217499 0.0196869996 -0.4994663883	---	0.4999833677 0.01999933471 -0.4995025246
4	X_1 X_2 X_3	0.4980316616 0.01992126647 -0.4995044772	---	---
5	X_1 X_2 X_3	0.4994955383 0.01997982153 -0.4995035371	---	---
6	X_1 X_2 X_3	0.4998614347 0.01999445739 -0.4995028027	---	---
7	X_1 X_2 X_3	0.4999528905 0.01999811562 -0.4995025953	---	---
8	X_1 X_2 X_3	0.4999757499 0.01999903 -0.4995025424	---	---
9	X_1 X_2 X_3	0.4999814637 0.01999925855 -0.4995025291	---	---
10	X_1 X_2 X_3	0.4999828918 0.01999931567 -0.4995025257	---	---
11	X_1 X_2 X_3	0.4999832488 0.01999932995 -0.4995025249	---	---
12	X_1 X_2 X_3	0.499983338 0.01999933352 -0.4995025247	---	---
13	X_1 X_2 X_3	0.4999833603 0.01999933441 -0.4995025246	---	---
14	X_1 X_2 X_3	0.4999833659 0.01999933463 -0.4995025246	---	---
15	X_1 X_2 X_3	0.4999833673 0.0199933469 -0.4995025246	---	---
16	X_1 X_2 X_3	0.4999833676 0.0199993347 -0.4995025246	---	---
17	X_1 X_2 X_3	0.4999833676 0.01999933471 -0.4995025246	---	---
CONCLUSION		CONVERGED IN 17 ITERATIONS	NO CONVERGENCE	CONVERGED IN 3 ITERATIONS

TABLE 3.2
For $x_1^{(0)} = 1$, $x_2^{(0)} = 1$, $x_3^{(0)} = 0$

Iteration Number	Variable	Iteration Method	Newton- Raphson Method	Parametric Method of Iteration
1	X_1 X_2 X_3	0.625 0.025 -0.475452213	0.5 0.5 -0.4867879441	0.5 0.02 -0.4995024917
2	X_1 X_2 X_3	0.5312323397 0.02124929359 -0.4982965416	0.4996539197 0.2503994463 -0.493806505	0.4999833666 0.01999933467 -0.4995025246
3	X_1 X_2 X_3	0.5077940706 0.02031176283 -0.4994302551	0.4999491556 0.1259982842 -0.4968557293	0.4999833677 0.01999933471 -0.4995025246
4	X_1 X_2 X_3	0.5019356544 0.02007742618 -0.4994953947	0.4999793144 0.06458633398 -0.4983882295	---
5	X_1 X_2 X_3	0.5004713422 0.02001885369 -0.4995012645	0.4999829242 0.0353895858 -0.4991178073	---
6	X_1 X_2 X_3	0.500105337 0.02000421348 -0.4995022346	0.49998333 0.02334579615 -0.4994188656	---
7	X_1 X_2 X_3	0.500013854 0.02000055416 -0.4995024533	0.4999833662 0.02023918093 -0.4994965282	---
8	X_1 X_2 X_3	0.4999909878 0.01999963951 -0.4995025069	0.4999833677 0.02000075587 -0.4995024891	---
9	X_1 X_2 X_3	0.4999852724 0.01999941089 -0.4995025202	0.4999833677 0.01999933476 -0.4995025246	---
10	X_1 X_2 X_3	0.4999838438 0.01999935375 -0.4995025235	0.4999833677 0.01999933471 -0.4995025246	---
11	X_1 X_2 X_3	0.4999834867 0.01999933947 -0.4995025243	---	---
12	X_1 X_2 X_3	0.4999833975 0.0199993359 -0.4995025245	---	---
13	X_1 X_2 X_3	0.4999833752 0.01999933501 -0.4995025246	---	---
14	X_1 X_2 X_3	0.4999833696 0.01999933478 -0.4995025246	--- ---	
15	X_1 X_2 X_3	0.4999833682 0.01999933473 -0.4995025246	---	---
16	X_1 X_2 X_3	0.4999833678 0.01999933471 -0.4995025246	---	---

CONCLUSION	CONVERGED IN 16 ITERATIONS	CONVERGED IN 10 ITERATIONS	CONVERGED IN 3 ITERATIONS	

TABLE 3.3
For $x_1^{(0)} = 5$, $x_2^{(0)} = 5$, $x_3^{(0)} = 5$

Iteration Number	Variable	Iteration Method	Newton- Raphson Method	Parametric Method of Iteration
1	X_1 X_2 X_3	1.622800703 0.06491202812 -0.2333342432	-1.257919398 2.493987329 -0.4500000001	0.497067604 0.01988270416 -0.4995082814
2	X_1 X_2 X_3	0.7806715004 0.03122686002 -0.4861548126	-0.1892295183 1.246638798 -1.343104476	0.4999835608 0.01999934243 -0.4995025242
3	X_1 X_2 X_3	0.5701390675 0.0228055627 -0.4987255541	0.751266281 0.6231139626 -0.5170993363	0.4999833677 0.01999933471 -0.4995025246
4	X_1 X_2 X_3	0.5175185969 0.02070074387 -0.4994318911	0.501323845 0.3117994466 -0.494510231	---
5	X_1 X_2 X_3	0.5043662885 0.02017465154 -0.4994908605	0.499950191 0.1565410286 -0.4961226146	---
6	X_1 X_2 X_3	0.5010788789 0.02004315515 -0.4994999011	0.4999754417 0.07954800913 -0.4980152391	---
7	X_1 X_2 X_3	0.5002571909 0.0200128764 -0.4995018832	0.4999824598 0.04228803314 -0.49894539	---
8	X_1 X_2 X_3	0.5000518099 0.0200020724 -0.499502365	0.4999832811 0.02587317067 -0.4993556857	---
9	X_1 X_2 X_3	0.5000004749 0.02000001899 -0.4995024848	0.4999833629 0.02066608603 -0.4994848556	---
10	X_1 X_2 X_3	0.4999876437 0.01999950575 -0.4995025147	0.4999833677 0.02001009043 -0.4995022556	---
11	X_1 X_2 X_3	0.4999844365 0.01999937746 -0.4995025221	0.4999833677 0.0199993376 -0.4995025245	---
12	X_1 X_2 X_3	0.4999836349 0.0199993454 -0.499502524	0.4999833677 0.01999933471 -0.4995025246	---
13	X_1 X_2 X_3	0.4999834345 0.01999933738 -0.4995025245	---	---
14	X_1 X_2 X_3	0.4999833844 0.01999933538 -0.4995025246	---	---
15	X_1 X_2 X_3	0.4999833719 0.01999933488 -0.4995025246	---	---
16	X_1 X_2 X_3	0.4999833688 0.01999933475 -0.4995025246	---	---

17	X_1 X_2 X_3	0.499983368 0.01999933472 -0.4995025246	---	---
18	X_1 X_2 X_3	0.4999833678 0.01999933471 -0.4995025246	---	---
CONCLUSION	CONVERGED IN 18 ITERATIONS	CONVERGED IN 12 ITERATIONS	CONVERGED IN 3 ITERATIONS	

Conclusion: From the above tabulated results it can be observed that the Parametric method of iteration looks more efficient than the Newton-Raphson method and Iteration method for the problems considered. It can also observed that Parametric method of Iteration converges even though Newton-Raphson method fails to converge from example (3.1).

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