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Restrained Triple Connected Domination Number of a Graph

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Abstract

The concept of triple connected graphs with real life application was introduced in [9] by considering the existence of a path containing any three vertices of a graph G. In[3], G. Mahadevan et. al., was introduced the concept of triple connected domination number of a graph. In this paper, we introduce a new domination parameter, called restrained triple connected domination number of a graph. A subset *S* of *V* of a nontrivial graph *G* is called a *dominating set* of *G* if every vertex in $V - S$ is adjacent to at least one vertex in *S*. The *domination number* $\gamma(G)$ of *G* is the minimum cardinality taken over all dominating sets in *G*. A subset *S* of *V* of a nontrivial graph *G* is called a *restrained dominating set* of *G* if every vertex in $V - S$ is adjacent to at least one vertex in *S* as well as another vertex in *V - S*. The *restrained domination number* $\gamma_r(G)$ of *G* is the minimum cardinality taken over all restrained dominating sets in *G*. A subset *S* of *V* of a nontrivial graph *G* is said to be triple connected dominating set, if *S* is a dominating set and the induced sub graph <*S*> is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by γ_{tc}. A subset *S* of *V* of a nontrivial graph *G* is said to be *restrained triple connected dominating set*, if *S* is a restrained dominating set and the induced sub graph <*S*> is triple connected. The minimum cardinality taken over all restrained triple connected dominating sets is called the *restrained triple connected domination number* and is denoted by γ_{rtc}. We determine this number for some standard graphs and obtain bounds for general graph. Its relationship with other graph theoretical parameters are also investigated.

Key words: Domination Number, Triple connected graph, Triple connected domination number, Restrained Triple connected domination number.
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1. Introduction

By a ***graph*** we mean a finite, simple, connected and undirected graph *G(V, E)*, where *V* denotes its vertex set and *E* its edge set. Unless otherwise stated, the graph *G* has *p* vertices and *q* edges. ***Degree*** of a vertex *v* is denoted by *d(v)*, the ***maximum degree*** of a graph *G* is denoted by $\Delta(G)$. We denote a ***cycle*** on *p* vertices by C_p, a ***path*** on *p* vertices by P_p, and a ***complete graph*** on *p* vertices by K_p. A graph *G* is ***connected*** if any two vertices of *G* are connected by a path. A maximal connected subgraph of a graph *G* is called a ***component*** of *G*. The number of components of *G* is denoted by *ω(G)*. The ***complement*** $\bar{G}$ of *G* is the graph with vertex set *V* in which two vertices are adjacent if and only if they are not adjacent in *G*. A ***tree*** is a connected acyclic graph. A ***bipartite graph*** (or ***bigraph***) is a graph whose vertex set can be divided into two disjoint sets V_1 and V_2 such that every edge has one end in V_1 and another end in V_2. A ***complete bipartite graph*** is a bipartite graph where every vertex of V_1 is adjacent to every vertex in V_2. The complete bipartite graph with partitions of order $|V_1|=m$ and $|V_2|=n$, is denoted by $K_{m,n}$. A ***star***, denoted by $K_{1,p-1}$ is a tree with one root vertex and p – 1 pendant vertices. A ***bistar***, denoted by *B(m, n)* is the graph obtained by joining the root vertices of the stars $K_{1,m}$ and $K_{1,n}$. The ***friendship graph***, denoted by F_n can be constructed by identifying *n* copies of the cycle C_3 at a common vertex. A ***wheel graph***, denoted by W_p is a graph with *p* vertices, formed by connecting a single vertex to all vertices of C_{p-1}. If *S* is a subset of *V*, then <*S*> denotes the vertex induced subgraph of *G* induced by *S*. The ***open neighbourhood*** of a set *S* of vertices of a graph G, denoted by *N(S)* is the set of all vertices adjacent to some vertex in *S* and $N(S) \cup S$ is called the ***closed neighbourhood*** of *S*, denoted by *N[S]*. A ***cut – vertex (cut edge)*** of a graph G is a vertex (edge) whose removal increases the number of components. A ***vertex cut***, or ***separating set*** of a connected graph *G* is a set of vertices whose removal results in a disconnected. The ***connectivity*** or ***vertex connectivity*** of a graph G, denoted by κ(*G*) (where G is not complete) is the size of a smallest vertex cut. The ***chromatic number*** of a graph *G*, denoted by χ(*G*) is the smallest number of colors needed to colour all the vertices of a graph *G* in which adjacent vertices receive different colours. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. A ***Nordhaus - Gaddum-type*** result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. Terms not defined here are used in the sense of [2].

A subset S of V is called a ***dominating set*** of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The ***domination number*** $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G. A dominating set S of a connected graph G is said to be a ***connected dominating set*** of G if the induced sub graph $<S>$ is connected. The minimum cardinality taken over all connected dominating sets is the ***connected domination number*** and is denoted by γ_c.

A dominating set is said to be **restrained dominating set** if every vertex in V – S is adjacent to atleast one vertex in S as well as another vertex in V - S. The minimum cardinality taken over all restrained dominating sets is called the **restrained domination number** and is denoted by γ_r.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [11, 12]. Recently, the concept of triple connected graphs has been introduced by J. Paulraj Joseph et. al.,[9] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graphs and established many results on them. A graph G is said to be ***triple connected*** if any three vertices lie on a path in G. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. In[3], G. Mahadevan et. al., was introduced the concept of triple connected domination number of a graph. A subset S of V of a nontrivial graph G is said to be a ***triple connected dominating set***, if S is a dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the ***triple connected domination number*** of G and is denoted by $\gamma_{tc}(G)$. Any triple connected dominating set with γ_{tc} vertices is called a γ_{tc} -set of G. In[4, 5, 6], G. Mahadevan et. al., was introduced **complementary triple connected domination number, complementary perfect triple connected domination number** and **paired triple connected domination number of a graph** and investigated new results on them.

In this paper, we use this idea to develop the concept of restrained triple connected dominating set and restrained triple connected domination number of a graph.

Theorem 1.1 [9] A tree T is triple connected if and only if $T \cong P_p;\ p \geq 3$.

Notation 1.2 Let G be a connected graph with m vertices $v_1, v_2, \ldots, v_m$. The graph obtained from G by attaching n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex v_2 and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \ldots, n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

Example 1.3 Let v_1, v_2, v_3, v_4, be the vertices of K_4. The graph $K_4(2P_2, 2P_2, 2P_3, P_3)$ is obtained from K_4 by attaching 2 times a pendant vertex of P_2 on v_1, 2 times a pendant vertex of P_2 on v_2, 2 times a pendant vertex of P_3 on v_3 and 1 time a pendant vertex of P_3 on v_4 and is shown in Figure 1.1.

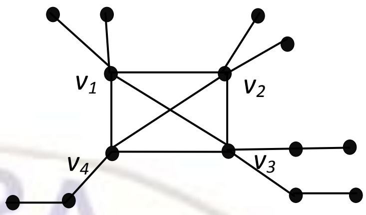

Figure 1.1 : $K_4(2P_2, 2P_2, 2P_3, P_3)$

2. Restrained Triple connected domination number

Definition 2.1 A subset S of V of a nontrivial graph G is said to be a *restrained triple connected dominating set*, if S is a restrained dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all restrained triple connected dominating sets is called the *restrained triple connected domination number* of G and is denoted by $\gamma_{rtc}(G)$. Any triple connected two dominating set with γ_{rtc} vertices is called a γ_{rtc} -set of G.

Example 2.2 For the graphs G_1, G_2, G_3 and G_4, in Figure 2.1, the heavy dotted vertices forms the restrained triple connected dominating sets.

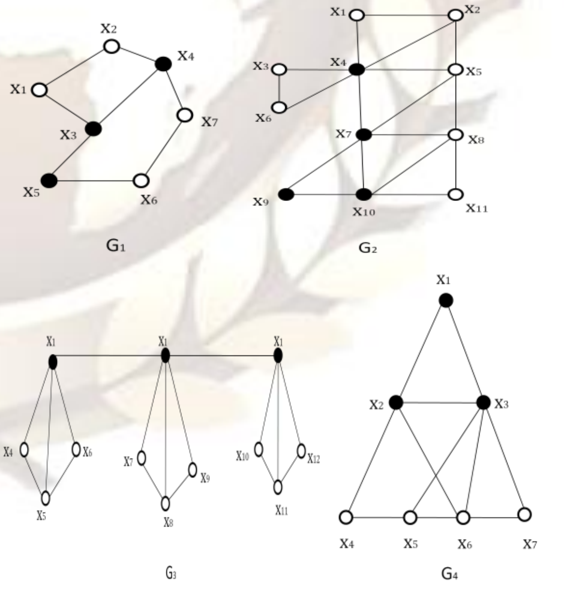

Figure 2.1 : Graph with $\gamma_{rtc} = 3$.

Observation 2.3 Restrained Triple connected dominating set (rtcd set) does not exists for all graphs and if exists, then $\gamma_{rtc}(G) \geq 3$.

Example 2.4 For the graph G_5, G_6 in Figure 2.2, we cannot find any restrained triple connected dominating set.

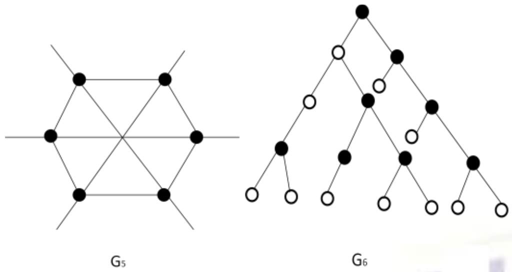

Figure 2.2 : Graphs with no rtcd set

Throughout this paper we consider only connected graphs for which triple connected two dominating set exists.

Observation 2.5 The complement of the restrained triple connected dominating set need not be a restrained triple connected dominating set.

Example 2.6 For the graph G_7 in Figure 2.3, $S = \{v_1, v_4, v_2\}$ forms a restrained triple connected dominating set of G_3. But the complement $V – S = \{v_3, v_5\}$ is not a restrained triple connected dominating set.

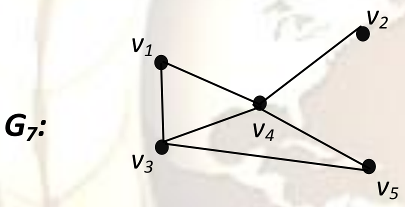

Figure 2.3 : Graph in which V–S is not a rtcd set

Observation 2.7 Every restrained triple connected dominating set is a dominating set but not conversely.

Observation 2.8 Every restrained triple connected dominating set is a connected dominating set but not conversely.

Exact value for some standard graphs:

1) For any cycle of order $p \geq 5$, $\gamma_{rtc}(C_p) = p - 2$.

2) For any complete graph of order $p \geq 5$, $\gamma_{rtc}(K_p) = 3$.

3) For any complete bipartite graph of order $p \geq 5$, $\gamma_{rtc}(K_{m,n}) = 3$.
(where $m, n \geq 2$ and $m + n = p$).

Observation 2.9 If a spanning sub graph H of a graph G has a restrained triple connected dominating set, then G also has a restrained triple connected dominating set.

Observation 2.10 Let G be a connected graph and H be a spanning sub graph of G. If H has a restrained triple connected dominating set, then $\gamma_{rtc}(G) \leq \gamma_{rtc}(H)$ and the bound is sharp.

Example 2.11 Consider the graph G_8 and its spanning subgraph H_8 and G_9 and its spanning subgraph H_9 shown in Figure 2.4.

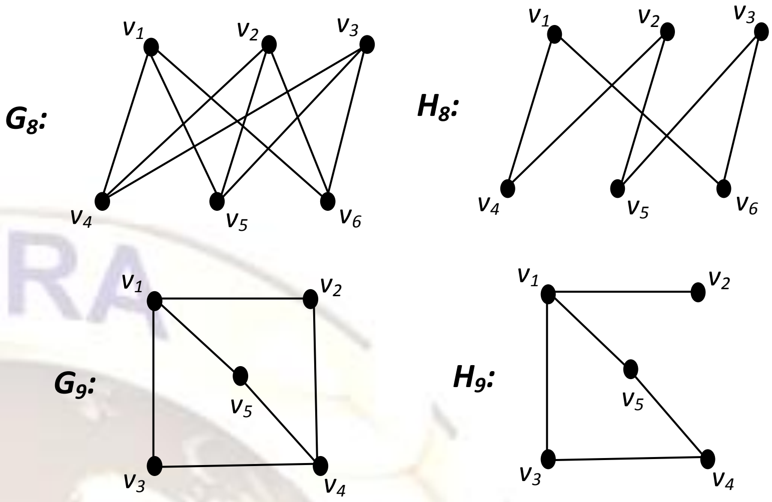

Figure 2.4

For the graph G_8, $S = \{v_1, v_4, v_2\}$ is a restrained triple connected dominating set and so $\gamma_{rtc}(G_4) = 3$. For the spanning subgraph H_8 of G_8, $S = \{v_1, v_4, v_2, v_5\}$ is a restrained triple connected dominating set so that $\gamma_{rtc}(H_8) = 4$. Hence $\gamma_{rtc}(G_8) < \gamma_{rtc}(H_8)$. For the graph G_9, $S = \{v_1, v_2, v_3\}$ is a restrained triple connected dominating set and so $\gamma_{rtc}(G_9) = 3$. For the spanning subgraph H_9 of G_9, $S = \{v_1, v_2, v_3\}$ is a restrained triple connected dominating set so that $\gamma_{rtc}(H_9) = 3$.Hence $\gamma_{rtc}(G_9) = \gamma_{rtc}(H_9)$.

Theorem 2.12 For any connected graph G with $p \geq 5$, we have $3 \leq \gamma_{rtc}(G) \leq p - 2$ and the bounds are sharp.

Proof The lower and bounds follows from *Definition 2.1*. For K_6, the lower bound is attained and for C_9 the upper bound is attained.

Theorem 2.13 For any connected graph G with 5 vertices, $\gamma_{rtc}(G) = p - 2$ if and only if $G \cong K_5, C_5, F_2, K_5 – e, K_4(P_2), C_4(P_2), C_3(P_3), C_3(2P_2)$ and any one of the following graphs given in Figure 2.5.

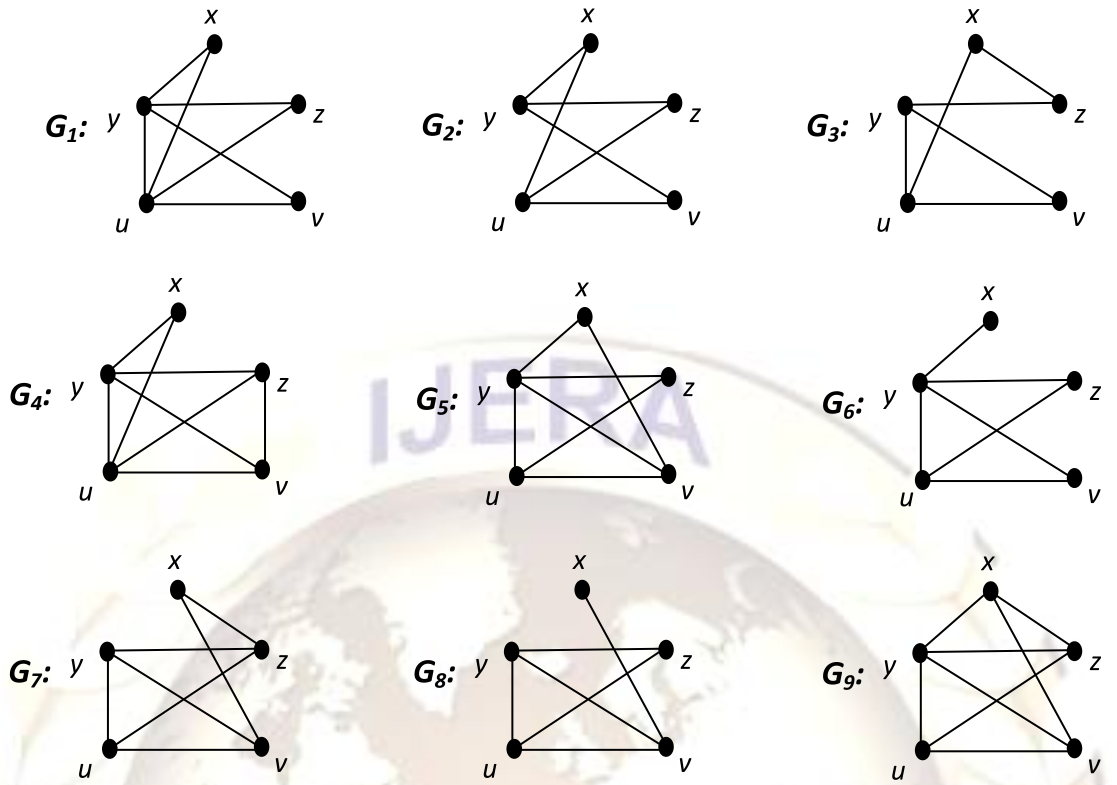

Figure 2.5

Proof Suppose G is isomorphic to K_5, C_5, F_2, $K_5 - e$, $K_4(P_2)$, $C_4(P_2)$, $C_3(P_3)$, $C_3(2P_2)$ and any one of the given graphs in Figure 2.5., then clearly $\gamma_{rtc}(G) = p - 2$. Conversely, Let G be a connected graph with 5 vertices, and $\gamma_{rtc}(G) = 3$. Let $S = \{x, y, z\}$ be the $\gamma_{rtc}(G)$ –set of G. Take $V - S = \{u, v\}$ and hence $<V - S> = K_2 = uv$.

Case (i) $<S> = P_3 = xyz$.

Since G is connected, x (or equivalently z) is adjacent to u (or equivalently v) (or) y is adjacent to u (or equivalently v). If x is adjacent to u. Since S is a restrained triple connected dominating set, v is adjacent to x (or) y (or) z. If v is adjacent to z, then $G \cong C_5$. If v is adjacent to y, then $G \cong C_4(P_2)$. Now by increasing the degrees of the vertices of $K_2 = uv$, we have $G \cong G_1$ to G_5, $K_5 - e$, $C_3(P_3)$. Now let y be adjacent to u. Since S is a restrained triple connected dominating set, v is adjacent to x (or) y (or) z. If v is adjacent to y, then $G \cong C_3(2P_2)$. If v is adjacent to y and z, x is adjacent to z, then $G \cong K_4(P_2)$. Now by increasing the degrees of the vertices, we have $G \cong G_6$ to G_8, $C_3(2P_2)$.

Case (ii) $<S> = C_3 = xyzx$.

Since G is connected, there exists a vertex in C_3 say x is adjacent to u (or) v. Let x be adjacent to u. Since S is a restrained triple connected dominating set, v is adjacent to x, then $G \cong F_2$. Now by increasing the degrees of the vertices, we have $G \cong G_9$, K_5. In all the other cases, no new graph exists.

The Nordhaus – Gaddum type result is given below:

Theorem 2.16 Let G be a graph such that G and $\bar{G}$ have no isolates of order $p \geq 5$. Then

(i) $\gamma_{rtc}(G) + \gamma_{rtc}(\bar{G}) \leq 2p - 4$

(ii) $\gamma_{rtc}(G).\ \gamma_{rtc}(\bar{G}) \leq (p-2)^2$ and the bound is sharp.

Proof The bound directly follows from *Theorem 2.12*. For cycle C_5, both the bounds are attained.

3 Relation with Other Graph Theoretical Parameters

Theorem 3.1 For any connected graph G with $p \geq 5$ vertices, $\gamma_{rtc}(G) + \kappa(G) \leq 2p - 3$ and the bound is sharp if and only if $G \cong K_5$.

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\kappa(G) \leq p - 1$ and by *Theorem 2.12*, $\gamma_{rtc}(G) \leq p - 2$. Hence $\gamma_{rtc}(G) + \kappa(G) \leq 2p - 3$. Suppose G is isomorphic to K_5. Then clearly $\gamma_{rtc}(G) + \kappa(G) = 2p - 3$. Conversely, Let $\gamma_{rtc}(G) + \kappa(G) = 2p - 3$. This is possible only if $\gamma_{rtc}(G) = p - 2$ and $\kappa(G) = p - 1$. But $\kappa(G) = p - 1$, and so $G \cong K_p$ for which $\gamma_{rtc}(G) = 3 = p - 2$. Hence $G \cong K_5$.

Theorem 3.2 For any connected graph G with $p \geq 5$ vertices, $\gamma_{rtc}(G) + \chi(G) \leq 2p - 2$ and the bound is sharp if and only if $G \cong K_5$.

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\chi(G) \leq p$ and by *Theorem 2.12*, $\gamma_{rtc}(G) \leq p - 2$. Hence $\gamma_{rtc}(G) + \chi(G) \leq 2p - 2$. Suppose G is isomorphic to K_5. Then clearly $\gamma_{rtc}(G) + \chi(G) = 2p - 2$. Conversely, let $\gamma_{rtc}(G) + \chi(G) =$

$2p - 2$. This is possible only if $\gamma_{rtc}(G) = p - 2$ and $\chi(G) = p$. Since $\chi(G) = p$, G is isomorphic to K_p for which $\gamma_{rtc}(G) = 3 = p - 2$. Hence $G \cong K_5$.

Theorem 3.3 For any connected graph G with $p \geq 5$ vertices, $\gamma_{rtc}(G) + \Delta(G) \leq 2p - 3$ and the bound is sharp.

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\Delta(G) \leq p - 1$ and by *Theorem 2.12*, $\gamma_{rtc}(G) \leq p$. Hence $\gamma_{rtc}(G) + \Delta(G) \leq 2p - 3$. For K_5, the bound is sharp.

REFERENCES

[1] Cokayne E. J. and Hedetniemi S. T. (1980): *Total domination in graphs*, Networks, Vol.10: 211–219.

[2] John Adrian Bondy, Murty U.S.R. (2009): *Graph Theory*, Springer, 2008.

[3] Mahadevan G., Selvam A., Paulraj Joseph J., and Subramanian T. (2012): *Triple connected domination number of a graph* , IJMC, Vol.3.

[4] Mahadevan G., Selvam A., Paulraj Joseph J., Ayisha B., and Subramanian T. (2012): *Complementary triple connected domination number of a graph* , *Accepted for publication in* Advances and Applications in Discrete Mathematics, ISSN 0974-1658.

[5] Mahadevan G, Selvam Avadayappan, Mydeen bibi A., Subramanian T. (2012): *Complementary perfect triple connected domination number of a graph*, International Journal of Engineering Research and Application, ISSN 2248-9622, Vol.2, Issue 5,Sep – Oct, pp 260-265.

[6] Mahadevan. G, Selvam Avadayappan, Nagarajan. A, Rajeswari. A, Subramanian. T. (2012): *Paired Triple connected domination number of a graph*, International Journal of Computational Engineering Research, *Vol. 2, Issue 5, Sep. 2012, pp. 1333-1338.*

[7] Nordhaus E. A. and Gaddum J. W. (1956): *On complementary graphs*, Amer. Math. Monthly, 63: 175–177.

[8] Paulraj Joseph J. and Arumugam. S. (1997): *Domination and coloring in graphs*, International Journal of Management Systems, 8 (1): 37–44.

[9] Paulraj Joseph J., Angel Jebitha M.K., Chithra Devi P. and Sudhana G. (2012): *Triple connected graphs*, Indian Journal of Mathematics and Mathematical Sciences, ISSN 0973-3329 Vol. 8, No. 1: 61-75.

[10] Paulraj Joseph J. and Mahadevan G. (2006): *On complementary perfect domination number of a graph*, Acta Ciencia Indica, Vol. XXXI M, No. 2.: 847–853.

[11] Sampathkumar, E.; Walikar, HB (1979): *The connected domination number of a graph*, *J. Math. Phys. Sci* 13 (6): 607–613.

[12] Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater (1998): *Domination in graphs*, Advanced Topics, Marcel Dekker, New York.

[13] Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater (1998): *Fundamentals of domination in graphs*, Marcel Dekker, New York.