Necessary and sufficient condition for the L^1-convergence of a modified trigonometric sum

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Abstract

We obtain a necessary and sufficient condition for L^1-convergence of a modified cosine sum which generalizes a result of Kumari and Ram [4]. We also deduce a result of Garrett and Stanojević [1] as a corollary of our result.

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1. Introduction

Definition 1.1. A null sequence $\{a_k\}$ is said to be of bounded variation BV if $\Sigma\, |\Delta a_k| < \infty$, where $\Delta a_k = a_k - a_{k+1}$.

Definition 1.2. A null sequence $\{a_k\}$ belongs to the class S[5] if there exists

a monotone sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$,

and $|\Delta a_k| \le A_k$, for all k .

Definition 1.3. A null sequence $\{a_k\}$ belongs to class C[1] if for every $\in > 0$, there exists $\delta > 0$, independent of n, and such that

$$C_n(\delta) = \int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k\, D_k(x) \right| dx < \in , \quad \text{for all } n ,$$

where $D_k(x)$ denotes Dirichlet's kernel.

Let us consider the cosine series

$$(1.1) \qquad f(x) = (a_0/2) + \sum_{k=1}^{\infty} a_k \cos kx ,$$

and let $S_n(x)$ denote the partial sum of (1.1) and $\lim_{n\to\infty} S_n(x) = f(x)$.

Regarding L^1-convergence of (1.1), Teljakovskii [6] proved the following result :

Theorem A. Let $\{a_k\}\ \varepsilon$ S . Then, $\| S_n(x) - f(x)\| = o(1)$, if and only if $a_n \log n = o(1)$, $n \to \infty$.

Garrett and Stanojević [1] extended Theorem A in the following manner :

Theorem B. Let $\{a_k\}\ \varepsilon$ BV $\cap$ C. Then $\|S_n(x) - f(x)\| = o(1)$, if and only if $a_n \log n = o(1)$, $n \to \infty$.

Later on, Garrett, Rees and Stanojević [2] proved that $S \subset C \cap BV$.

Kumari and Ram [4] introduced a new modified cosine sum

$$(1.2) \qquad f_n(x) = (a_0/2) + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j)\, k \cos kx,$$

and proved the following result :

Theorem C. Let $\{a_k\}\ \varepsilon$ S and $a_n \log n = o(1)$, $n \to \infty$, then

$$\|f(x) - f_n(x)\| = o(1) , \quad n \to \infty .$$

In this paper, we obtain a necessary and sufficient condition for the L^1-convergence of the limit of (1.2) which generalizes Theorem C and deduce Theorem B as a corollary from our result.

2. Lemma

We require the following lemma for the proof of our theorem :

Lemma 1[3]. Let $D_n(x)$, $\bar{D}_n(x)$ and $F_n(x)$ denote Dirichlet, conjugate Dirichlet and Fejér kernels respectively, then

$$F_n(x) = D_n(x) - [1/(n+1)]\, \bar{D}'_n(x) .$$

3. Result

We shall prove

Theorem 1. Let $\{a_k\}\ \varepsilon$ BV $\cap$ C, then $\|f(x) - f_n(x)\| = o(1)$, if and only if $a_n \log n = o(1)$, $n \to \infty$.

Proof. Using Lemma 1 and summation by parts, we get

$$(3.1) \qquad f_n(x) = (a_0/2) + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j)\, k \cos kx$$

$$= (a_0/2) + \sum_{k=1}^{n} a_k \cos kx - (a_{n+1} / (n+1))\, \bar{D}'_n (x) .$$

$$= (a_0/2) + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) + a_{n+1} F_n(x)$$

$$= S_n(x) - a_{n+1} D_n(x) + a_{n+1} F_n(x)$$

$$= \sum_{k=0}^{n-1} \Delta a_k\, D_k(x) + \Delta a_n D_n(x) + a_{n+1} F_n(x) .$$

Now,

$$\int_0^\pi | f(x) - f_n(x) | \, dx = \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) - \Delta a_n D_n(x) - a_{n+1} F_n(x) \right| dx$$

$$\le \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^\pi |\Delta a_n D_n(x)| dx + |a_{n+1}| \left| \int_0^\pi F_n(x) \, dx \right|$$

$$= \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx + \int_\delta^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^\pi |\Delta a_n D_n(x)| dx + |a_{n+1}| (\pi/2)$$

$$\le (\in/4) + \sum_{k=n}^{\infty} |\Delta a_k| \int_\delta^\pi \csc (x/2) \, dx + |\Delta a_n| \int_0^\pi |D_n(x)| dx + (\in/4)$$

$$= (\in/4) + \sum_{k=n}^{\infty} |\Delta a_k| \left[-2 \log |\csc (\delta/2) - \cot (\delta/2)| \right] + |\Delta a_n| \int_0^\pi |D_n(x)| dx + (\in/4) < \in ,$$

since $\int_0^\pi |D_n(x)| \, dx$ behaves like $\log n$ for large n and $\{a_k\} \,\varepsilon\, BV \cap C$. Also, using (3.1)

$$\int_0^\pi |a_{n+1} D_n(x)| \, dx \le \int_0^\pi |S_n(x) - f_n(x)| dx + |a_{n+1}| \int_0^\pi F_n(x) \, dx$$

$$\le \int_0^\pi |f(x) - f_n(x)| dx + (\pi/2) a_{n+1} ,$$

This completes our result.

Corollary 1. Let $\{a_k\} \,\varepsilon\, BV \cap C$, then $\|S_n(x) - f(x)\| = o(1)$, if and only if $a_n \log n = o(1)$, $n \to \infty$.

Proof. We have

$$\int_0^\pi |f(x) - S_n(x)| dx = \int_0^\pi |f(x) - f_n(x) + f_n(x) - S_n(x)| dx$$

$$\le \int_0^\pi |f(x) - f_n(x)| dx + \int_0^\pi |f_n(x) - S_n(x)| \, dx$$

$$\le \int_0^\pi |f(x) - f_n(x)| dx + \int_0^\pi | a_{n+1} D_n(x) | \, dx + \int_0^\pi |a_{n+1} F_n(x)| dx$$

$$\le \int_0^\pi |f(x) - f_n(x)| dx + \int_0^\pi |a_{n+1} D_n(x) | \, dx + (\pi/2) a_{n+1} .$$

Also,

$$\int_0^\pi |a_{n+1} D_n(x)| dx \le \int_0^\pi |f_n(x) - S_n(x)| dx + \int_0^\pi |a_{n+1} F_n(x)| \, dx$$

$$\le \int_0^\pi |f(x) - S_n(x)| \, dx + (\pi/2) a_{n+1} .$$

Since $\int_0^\pi |a_{n+1} D_n(x)| \, dx$ behaves like $a_n \log n$ for large values of n, and by our theorem $\lim_{n\to\infty} \int_0^\pi |f(x) - f_n(x)| = 0$, Theorem B of Garrett and Stanojević follows.

References

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