Analytic Representations Of Convex And Starlike Functions

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ABSTRACT

In this paper we investigated the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In particular, we consider the class G_b consisting of normalized functions f. We determine values of b for which $G_b \subset S^*(\alpha)$,$1/2 \le \alpha < 1$ and also find values of b for which $G_b \subset K$. It is known that $K \subset S^*(1/2)$, showing that $G_1 \subset S^*(1/2) - K$. We also find values of b for which G_b is not starlike and not univalent.

Key Words: Univalent Function, Starlike Function, convex Function, Analytic Function

1. INTRODUCTION:

A function f of the complex variable 'z' is analytic in an open disc if it is analytic at each point in that disc.

An analytic function f on a domain D is said to be univalent if it does not take the same value twice i.e., $f(z_1) \ne f(z_2)$ for all pairs of distinct points z_1 and z_2 in D.

A conformal mapping of the unit disc onto a domain starlike with respect to the origin is said to be starlike function.

Let S denote the class of functions f normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. A function f in S is said to be starlike of order $\alpha, 0 \le \alpha < 1$, and is denoted by S* (α) if Re $\{zf'(z)/f(z)\} > \alpha$, $z \in \Delta$ and is said to be convex and is denoted by K if Re $\{1 + zf''(z)/f'(z)\} > 0,\ z \in \Delta$. Mocanu studied linear combinations of the representations of convex and starlike functions and defined the class of α-convex functions. S.S. Miller was shown that if $\mathrm{Re}\,\{\alpha(1 + zf''(z)/f'(z)) + (1-\alpha)zf'(z)/f(z)\} > 0$ for $z \in \Delta$, then f is starlike for α real and convex for $\alpha \ge 1$.

In this paper we investigated the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In particular, we consider the class G_b consisting of normalized functions f defined by

$$G_b = \left\{ f : \left| \left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \right) - 1 \right| < b, z \in \Delta \right\}$$

We determine values of b for which $G_b \subset S^*(\alpha)$,$1/2 \le \alpha < 1$ and also find values of b for which $G_b \subset K$ It is known that $K \subset S^*(1/2)$. Show that $G_1 \subset S^*(1/2) - K$ we also find values of b for which G_b is not starlike and not univalent.

Let T (P) denote the class of function f(z) of the form

$$f(z) = z^p + \sum_{k=p+1}^{\varepsilon} a_k z^k \quad (p \in n = \{1, 2, 3\text{--------}\})$$

Which are analytic and P-valent in the open unit disk $U = \{z : z \in c \text{ and } |z| < 1\}$

In this paper D_z^q denotes the q[th]- order differential operator, for a function $f(z) \in T(P)$.

$$D_z^q f(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\alpha} \frac{k!}{(k-q)!} a_k z^{k-q}$$

$$(p > q;\ p \in n : q \in n_0)$$

To prove our result, we need the following Lemma given by jack.

1.2. Lemma: Suppose 'ω' is analytic for $|z| \le r, \omega(0) = 0$ and $|\omega(z_0)| = \underset{|z| = r}{Max} |\omega(z)|$

then $z_0 \omega'(z_0) = k\omega(z_0), k \ge 1.$

1.3. Theorem: Let $b = [1 - \alpha(p,q)]/2[\alpha(p,q)]^2$, $\frac{1}{2} \le \alpha(p,q) < 1$. Then $G_b \subset S^*[\alpha(p,q)]$, with extremal function $z/(1-z)^{2[1-\alpha(p,q)]}$.

Proof: It is well known that if $\omega(z)$ is analytic in Δ with $\omega(0) = 0$, then $\mathrm{Re}\left[\frac{1 + [1 - 2\alpha(p,q)]\omega(z)}{1 - \omega(z)}\right] > \alpha$ (p,q), $z \in \Delta$, if and only if $\omega(z)$ is a Schwarz function, i.e. $|\omega(z)|<1$ for $z\in\Delta$ with $\omega(0)=0$.

Let p (z) be an analytic function defined by

(1.3.1)

$$p(z)=\frac{zf'(z)}{f(z)\,(p-q)}=\frac{1+[1-2\alpha(p,q)]\omega(z)}{1-\omega(z)}$$

Differences on both sides we get

$$p'(z)=\frac{[f(z)(p-q)][f'(z)+zf''(z)]-[zf'(z)f'(z)(p-q)]}{[f(z)]^2\,(p-q)^2}$$

$$\frac{zp'(z)}{p(z)}=z\left[\frac{[f(z)(p-q)][f'(z)+zf''(z)]-[zf'(z)f'(z)(p-q)]}{[f(z)]^2\,(p-q)^2}\right]\times\frac{f(z)(p-q)}{zf'(z)}$$

$$=\frac{[f(z)(p-q)]\,[f'(z)+zf''(z)]-[zf'(z)f'(z)(p-q)]}{f'(z)\backslash\cdot f(z)\,(p-q)}$$

$$\frac{zp'(z)}{p(z)}=\frac{f(z)(p-q)zf''(z)+f(z)(p-q)f'(z)-f'(z)(p-q)zf'(z)}{f'(z)\cdot f(z)\cdot(p-q)}$$

$$\frac{zp'(z)}{p(z)}=\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}$$

$$1+\frac{zf''(z)}{f'(z)}=\frac{zp'(z)}{p(z)}+\frac{zf'(z)}{f(z)}$$

$$1+\frac{zf''(z)}{f'(z)}=\frac{zp'(z)}{p(z)}+p(z)(p-q)$$

$$\left\{\left[\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right]-1\right\}=\frac{\frac{zp'(z)}{p(z)}+p(z)(p-q)}{p(z)(p-q)}-1$$

$$=\left[\frac{zp'(z)}{[p(z)]^2(p-q)}+1\right]-1$$

$$=\frac{zp'(z)}{[p(z)]^2(p-q)}$$

and

(1.3.2)

$$\frac{zp'(z)}{[p(z)]^2(p-q)}=\left|\frac{2[1-\alpha(p,q)]z\omega'(z)}{[1+\{1-2[\alpha(p,q)]\}\omega(z)]^2}\right|$$

$$\left\{\left[\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right]-1\right\}=$$

$$\frac{zp'(z)}{[p(z)]^2(p-q)}=\left|\frac{2[1-\alpha(p,q)]z\omega'(z)}{[1+\{1-2[\alpha(p,q)]\}\omega(z)]^2}\right|$$

If $f\notin S^*\alpha(p,q)$, then by Lemma 1.2 there is a $z_0\in\Delta$ for which $|\omega(z_0)|=1$ and $z_0\omega'(z_0)\geq\omega(z_0)$. It follows from (1.3.2) that $\left|\frac{z_0p'(z_0)}{(p-q)[p(z_0)]^2}\right|\geq\frac{2[1-\alpha(p,q)]}{[2\alpha(p,q)]^2}$

Which contradicts our hypothesis.

This completes the proof of the Theorem.

By taking p=1, q=0, we have the following corollary which states as follows

1.4. Corollary: If

$$p(z)=\frac{zf'(z)}{f(z)}=\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}$$

then

$$1=\frac{zf''(z)}{f'(z)}=p(z)+\frac{zp'(z)}{p(z)}$$

and

$$\left|\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right)-1\right|=\left|\frac{zp'(z)}{[p(z)]^2}\right|=\left|\frac{2(1-\alpha)zw'(z)}{[1+(1-2\alpha)w(z)]^2}\right|$$

By above theorem, we have the following corollary

1.5. Corollary: $G_1\subset S^*(1/2)$

Proof: By putting b=1, p=1, q=0 in Theorem 1.3 we get the proof of the corollary.

1.6. Theorem: If $f\in S^*(1/2)$, then $\left|\left[\frac{1+zf''(z)/f'(z)(p-q)}{zf'(z)/f(z)(p-q)}\right]-1\right|<1$ for $|z|<(2\sqrt{3}-3)^{1/2}=0.68$----

Proof: Let $P(z)=\frac{zf'(z)}{(p-q)f(z)}=\frac{1}{1-w(z)}$

Where w (z) is a Schwarz function. We need to find the largest disk $|z|<R$ for which

$$\left|\frac{zp'(z)}{[p(z)]^2(p-q)}\right|=|zw'(z)|<1.$$

Dieudonne found the region of values for the derivatives of Schwarz functions. This led to the sharp bound.

(1.6.1)

$$|w'(z)| \le \begin{cases} 1, & r = |z| \le \sqrt{2}-1 \\ \dfrac{(1+r^2)^2}{4r(1-r^2)}, & r \ge \sqrt{2}-1 \end{cases}$$

Since $|zw'(z)| \le \dfrac{(1+r^2)^2}{4(1-r^2)} = 1$ for $r = (2\sqrt{3}-3)^{\frac{1}{2}}$

Which completes the proof of the theorem.

1.7. Theorem: $G_1 \not\subset K$

Proof: $G_1 \subset S^*(1/2)$, for $f \in G_1$ satisfies

$$\frac{zf'(z)}{(p-q)f(z)} = \frac{1}{1-w(z)}$$

for some Schwarz function w(z).

Putting α =1/2 in (1.3.2) we get $f \in G_1 \Leftrightarrow |zw'(z)| < 1$ for $z \in \Delta$, which means that $zw'(z)$ also be a Schwarz function.

Since $\dfrac{1+zf''(z)}{(p-q)f(z)} = \dfrac{zw'(z)}{1-w(z)}$ is sufficient to construct a Schwarz function $\Omega(z) = zw'(z)$ for which

(1.7.1) $$\mathrm{Re}\left\{\frac{1+\Omega(z)}{1-w(z)}\right\} < 0$$

At some point $z \in \overline{\Delta}$.

Let

$$A = \left\{ z \in \Delta : |z - z_0| < 10^{-5},\ z_0 = e^{\frac{\pi i}{4}} = e^{i\theta_0} \right\}$$

And $\phi(z) = (z_0 + \overline{z_0})\left[(1-\overline{z_0}z)^{\frac{1}{n}} - 1\right]$

Where n is large enough so that $|\phi(z)/z| < 10^{-4}$

for $z \in \Delta - A$ and $|\mathrm{Im}\phi(z)| < 10^{-8}$

for $z \in A$. Define Ω by Ω (z) = 0.9999 $[z + \phi(z)]$.

We first show that Ω (z) is a Schwarz function and then show that the inequality (1.7.1) holds when $z = z_0$.

If $z \in \Delta - A$

Then

$$|\Omega(z)| \le 0.9999(|z| + |\phi(z)|) \le 0.9999(1.0001) < 1$$

If $z \in A$, $z = z_0 - \varepsilon e^{i\beta}$, $0 < \varepsilon < 10^{-5}$, and note that $-2\cos\theta_0 \le \mathrm{Re}\,\phi(z) \le 0$.

If $\mathrm{Re}(z + \phi(z)) \ge 0$, then

$|z + \mathrm{Re}\,\phi(z)| \le |z| < 1$.

If $\mathrm{Re}(z + \phi(z)) < 0$, then

$$|z + \mathrm{Re}\,\phi(z)| \le \sqrt{(\cos\theta_0 + \varepsilon)^2 + (\sin\theta_0 + \varepsilon)^2} < \sqrt{1+4\varepsilon} < 1 + 2\varepsilon < 1.0001$$

Thus, If $z \in A$,

$$|\Omega(z)| \le 0.9999|z + \mathrm{Re}\,\phi(z)| + |\mathrm{Im}\phi(z)| < 0.9999(1.0001) + 10^{-8} = 1.$$

Therefore, $\Omega(z)$ is a Schwarz function.

Now we show that (1.7.1) holds at $z = z_0$ for $\Omega(z)$.

Since

$$\left|\frac{\Omega(z)}{z} - 1\right| = |w'(z)| < 0.0002 \quad for \quad z \in \Delta - A$$

We write $w(z) = z + \eta(z)$, where $|\eta(z)| < 0.0003$ for $z \in A$.

Note that

$$\left(|1 - w(z_0)|^2\right)\mathrm{Re}\left(\frac{1+\Omega(z_0)}{1-\Omega(z_0)}\right) = \mathrm{Re}\left\{(1-\Omega(z_0))\left(1 + \overline{w(z_0)}\right)\right\}$$

$$= \mathrm{Re}\left\{\left(1 - 0.9999\overline{z_0}\right)\left(1 - \overline{z_0} - \eta\overline{(z_0)}\right)\right\}$$

$$\le 1 - 1.9999\cos\theta_0 + 0.9999\cos 2\theta_0 + 2|\eta(z_0)|$$

$$< 1 - 1.9999\cos\left(\pi/4\right) + 00006 < 0.$$

Hence, the function f for which $\dfrac{1+zf''(z)}{f'(z)(p-q)} = \dfrac{1+\Omega(z)}{1-w(z)}$ must be in $G_1 - k$.

Which completes the proof of the theorem.

1.8. Theorem: $G_b \subset k$ for $b \le \sqrt{2}/2$.

Proof: Since $f \in G_b \subset G_1 \subset S^*(1/2)$.

We write $\dfrac{zf'(z)}{f(z)(p-q)} = \dfrac{1}{1-w(z)}$

Where w is a Schwarz function. For $f \in G_b$, we take $\alpha = 1/2$ in (1.3.2) to obtain $|zw'(z)| < \sqrt{2}/2$ and $|w(z)| < \sqrt{2}/2$, $z \in \Delta$.

We have to show that

(1.8.1)

$$\mathrm{Re}\left\{\frac{1+zf''(z)}{f'(z)(p-q)}\right\}=\mathrm{Re}\left\{\frac{1+zw'(z)}{1-w(z)}\right\}>0.$$

$$\left|\arg\left(\frac{1+zw'(z)}{1-w(z)}\right)\right|\le|\arg(1+zw'(z))|+|\arg(1-w(z))|$$

$$\le\frac{\pi}{4}+\frac{\pi}{4}$$

$$=\frac{\pi}{2}$$

the result follows.

Mac Gregor found the radius of convexity for $S^*(1/2)$ to be $\left(2\sqrt{3}-3\right)^{1/2}=0.68$ ----------- Since $G_1\subset S^*(1/2)$ we know that the radius of convexity is at least this large.

1.9. Corollary: If $f\in G_b$, $\sqrt{2}/2\le b\le 1$, then f is convex in the disk $|z|<\sqrt{2}/2b$.

Proof: If $|zw'(z)|<1$ for $z\in\Delta$, then $|zw'(z)|<t$ for $|z|<t<1$.

If $f\in G_b$, then $|zw'(z)|<b$ for $z\in\Delta$. Hence $|zw'(z)|<\sqrt{2}/2$, then $|z|<\sqrt{2}/2b$.

Now we illustrate this by examples

1.10. Examples: Theorem 1.3 gives order of starlikeness for G_b when $0<b\le 1$, with $G_1\subset S^*(1/2)$. Our method do not extend to b>1, but we expect the order of starlikeness to decrease from ½ to '0' as 'b' increase from '1' to some value b_0 after which functions in G_b need not be starlike. We do not have a sharp result for b>1, but the next example shows that the univalent functions in G_b are not necessarily starlike for b≥11.66.

The function $h(z)=z(1-iz)^{i-1}$ is spiral-like and hence, in S because

(1.10.1)

$$\mathrm{Re}\left\{e^{\pi i/4}\frac{zh'(z)}{h(z)}\right\}=\frac{1}{\sqrt{2}}\left(\frac{1-|z|^2}{|1-iz|^2}\right)>0,\qquad z\in\Delta$$

Since $\dfrac{zh'(z)}{h(z)}=\dfrac{(1+z)}{(1-iz)}$,

We see that h is not starlike for $|z|<a$, $\sqrt{2}/2<a<1$.

Thus, $f(z)=f_a(z)=h(az)/a$ is not starlike for $z\in\Delta$.

By putting

$$p(z)=zf'(z),\qquad f(z)=\frac{(1+az)}{(1-aiz)},$$

We have

(1.10.2)

$$\left|\frac{zp'(z)}{(p(z))^2}\right|=\left|\frac{(1+i)az}{(1+az)^2}\right|\le\frac{\sqrt{2}a}{(1-a)^2}<11.66$$

for a sufficiently close to $\sqrt{2}/2$. Hence, $f\in G_b-S^*(0)$ for b=11.66.

We show that the function in G_b need not be univalent. It is shown for $h(z)=z(1-iz)^{i-1}$ then $g(z)=\int_0^z\frac{h(t)}{t}dt=(1-iz)^i-1$ is not in S because $g(z_0)=g(-z_0)$ for $z_0=i(e^{2\pi}-1)/(e^{2\pi}+1)$, $|z_0|=0.996$------ we conclude that for f(z)=g(cz)/c, c=0.997, 4 for 'b' sufficiently large.

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