On πgb-D-sets and Some Low Separation Axioms

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Abstract
This paper introduces and investigates some weak separation axioms by using the notions of πgb-closed sets. Discussions has been carried out on its properties and its various characterizations.

Mathematics Subject Classification: 54C05

Keywords: πgb-R, πgb-D, πgb-D-connected, πgb-D-compact.

1.Introduction
Levine [16] introduced the concept of generalized closed sets in topological space and a class of topological spaces called T ½ spaces. The investigation of generalized closed sets leads to several new separation axioms. Andrijevic [3] introduced a new class of generalized open sets in a topological space, the so-called b-open sets. This type of sets was discussed by Ekici and Caldas [11] under the name of γ-open sets. The class of b-open sets is contained in the class of semi-pre-open sets and contains all semi-open sets and pre-open sets. The class of b-open sets generates the same topology as the class of pre-open sets. Since the advent of these notions, several research paper with interesting results in different respects came to existence[1,3,7,11,12,20,21,22]. Extensive research on generalizing closedness was done in recent years as the notions of a generalized closed, generalized semi-closed, α-generalized closed, generalized semi-pre-open closed sets were investigated in [2,8,16,18,19]. In this paper, we have introduced a new generalized axiom called πgb-separation axioms. We have incorporated πgb-D, πgb-R, spaces and a study has been made to characterize their fundamental properties.

2. Preliminaries
Throughout this paper (X, τ) and (Y, τ) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A respectively. (X, τ) will be replaced by X if there is no chance of confusion.

Let us recall the following definitions which we shall require later.

Definition 2.1: A subset A of a space (X, τ) is called (1) a regular open set if A = int(cl(A)) and a regular closed set if A = cl(int(A)); (2) b-open [3] or sp-open [9], γ -open [11] if A ⊂ cl(int(A))∪int(cl(A)). The complement of a b-open set is said to be b-closed [3]. The intersection of all b-closed sets of X containing A is called the b-closure of A and is denoted by bCl(A). The union of all b-open sets of X contained in A is called b-interior of A and is denoted by bInt(A). The family of all b-open (resp. α-open, semi-open, preopen, β-open, b-closed, preclosed) subsets of a space X is denoted by bO(X)(resp. αO(X), SO(X), PO(X), βO(X), bC(X), PC(X)) and the collection of all b-open subsets of X containing a fixed point x is denoted by bO(X,x). The sets SO(X, x), αO(X, x), PO(X, x), βO(X, x) are defined analogously.

Lemma 2.2 [3]: Let A be a subset of a space X. Then (1) bCl(A) = sCl(A) ∩ Cl(int(A)) ∩ Cl(int(A)); (2) bInt(A) = sInt(A) ⊂ pInt(A) = A ∩ [Int(Cl(A)) ∪ Cl(int(A))];

Definition 2.3: A subset A of a space (X, τ) is called (1) a generalized b-closed (briefly gb-closed)[12] if bcl(A) ⊂ U whenever A ⊂ U and U is open. 2) πg-closed [10] if cl(A)⊂U whenever A⊂U and U is π-open. 3) πgb -closed [23] if bcl(A)⊂U whenever A⊂U and U is π-open in (X, τ). By πGBG(τ) we mean the family of all πgb- closed subsets of the space(X, τ).

Definition 2.4: A function f: (X, τ) → (Y, σ) is called (1) πgb- continuous [23] if every f¬(V) is πgb- closed in (X, τ) for every closed set V of (Y, σ). 2) πgb- irresolute [23] if f¬(V) is πgb- closed in (X, τ) for every πgb-closed set V in (Y, σ).

Definition[24]: (X, τ̄) is πgb-T0 if for each pair of distinct points x, y of X, there exists a πgb -open set containing one of the points but not the other.

Definition[24]: (X, τ) is πgb-T1 if for any pair of distinct points x, y of X, there is a πgb -open set U in X such that x ∈ U and y ∉ U and there is a πgb -open set V in X such that y ∈ U and x ∉ V.

Definition[24]: (X, τ) is πgb-T2 if for each pair of distinct points x and y in X, there exists a πgb -open set U and a πgb -open set V in X such that x ∈ U, y ∈ V and U ∩ V = ∅.
Definition: A subset A of a topological space (X, τ) is called:
(i) D-set [25] if there are two open sets U and V such that U ≠ X and A = U - V.
(ii) sD-set [5] if there are two semi-open sets U and V such that U ≠ X and A = U - V.
(iii) pD-set [14] if there are two preopen sets U and V such that U ≠ X and A = U - V.
(iv) aD-set [6] if there are two U, V ∈ αO(X, τ) such that U ≠ X and A = U - V.
(v) bD-set [15] if there are two U, V ∈ BO(X, τ) such that U ≠ X and A = U - V.

Definition 2.6[17]: A subset A of a topological space X is called an g̃D-set if there are two g̃α, open sets U, V such that U ≠ X and A = U - V.

Definition 2.7[4]: X is said to be (i)rgα-Rα iff rga-{x} ∈ G whenever x ∈ G ∈ RGoO(X).

Definition 13: A topological space (X, τ) is said to be D-compact if every cover of X by D-sets has a finite subcover.

Definition 15: A topological space (X, τ) is said to be bD-compact if every cover of X by bD-sets has a finite subcover.

3. πgbd-sets and associated separation axioms

Definition 3.1: A subset A of a topological space X is called πgbd-set if there are two U, V ∈ πGBO(X, τ) such that U ≠ X and A = U - V.

Example 3.2: Let X = {a, b, c} and τ = {Φ, {a}, {b}, {a, b}, X}. Then {c} is a πgbd-set but not πgbd-open since πGBO(X, τ) = {Φ, {a}, {b}, {a, b}, X}. Then U = {b, c} ≠ X and V = {a, b} are πgbd-open sets in X. For U and V, since U - V = {b, c} - {a, b} = {c}, then we have S = {c} is a πgbd-set but not πgbd-open.

Theorem 3.3: Every D-set, aD-set, pD-set, bD-set, sD-set is a πgbd-set.

Converse of the above statement need not be true as shown in the following example.

Example 3.4: Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, X}. πGBO(X, τ) = P(X). Hence πgbd-set = P(X). {b, c, d} is a πgbd-set but not D-set, aD-set, pD-set, bD-set, sD-set.

Definition 3.5: X is said to be
(i) πgbd-Di if for any pair of distinct points x and y of X, there exist a πgbd-D-set in X containing x but not y (or a πgbd-D-set in X containing y but not x).
(ii) πgbd-D1 if for any pair of distinct points x and y in X, there exists a πgbd-D-set of X containing x but not y and a πgbd-D-set in X containing y but not x.
(iii) πgbd-D2 if for any pair of distinct points x and y of X, there exists disjoint πgbd-D-sets G and H in X containing x and y respectively.

Example 3.6: Let X = {a, b, c, d} and τ = {Φ, {a}, {a, b}, {c, d}, {a, c, d}, X}. Then X is πgbd-D1, i = 0, 1, 2.

Remark 3.7
(i) If (X, τ) is πgbd-D1 then (X, τ) is πgbd-D2.
(ii) If (X, τ) is πgbd-D2 then it is πgbd-D1.
(iii) If (X, τ) is πgbd-D1 then it is πgbd-D2.

Theorem 3.8: For a topological space (X, τ), the following statements hold.
(i) (X, τ) is πgbd-D1 if it is πgbd-D2.
(ii) (X, τ) is πgbd-D2 if it is πgbd-D1.

Proof: (i) The sufficiency is stated in remark 3.7(i).

(ii) Necessity: Suppose X is πgbd-D1. Then for any two distinct points x, y ∈ X, there exist open sets U, V such that x ∈ U, y ∈ V, and U ∩ V = Φ. Let G be the set of all such open sets U. Then G is a πgbd-D-set.

(2) Sufficiency: Remark 3.7(ii).

Necessity: Suppose X is πgbd-D1. Then for each distinct pair x, y ∈ X, we have πgbd-D-sets G1 and G2 such that x ∈ G1 and y ∈ G2. Let G be the set of all such pairs (x, y). Then G is a πgbd-D-set.

Now we have two cases: (i) x ∈ U or y ∈ U and (ii) x ∉ U or y ∉ U. In case (i), if x ∈ U and y ∈ U, then we have πgbd-D1 and if x ∉ U or y ∉ U, then we have πgbd-D2.
Proof: Remark 3.7 and theorem 3.8

Definition 3.10: Let (X,τ) be a topological space. Let x be a point of X and G be a subset of X. Then G is called an ngb-neighbourhood of x (briefly ngb- nh of x) if there exists an ngb-open set U of X such that x ∈ U ⊆ G.

Definition 3.11: A point x ∈ X which has X as a ngb-neighbourhood is called ngb-neat point.

Example 3.12: Let X = {a, b, c}, τ = {Φ, {a}, {b}, {a, b}, X}. ngbBO(X, τ) = {{Φ, {a}, {b}, {a, b}, {a, c}, X}. The point {c} is a ngb-neat point.

Theorem 3.13: For a ngb-T0 topological space (X, τ), the following are equivalent.
(i) (X, τ) is a ngb-D1
(ii) (X, τ) has no ngb-neat point.

Proof: (i) ⇒ (ii). Since X is a ngb-D1, then each point x of X is contained in an ngb-D-set O = U ∪ V and hence in U. By definition, U ∪ V is X. This implies x is not a ngb-neat point.

(ii) ⇒ (i). If X is ngb-T0, then for each distinct points x, y ∈ X, at least one of them say(x) has a ngb-neighbourhood U containing x and not y. Thus U ∩ [X \ U] is a ngb-D-set. If X has no ngb-neat point, then y is not a ngb-neat point. That is there exists ngb-neighbourhood V of y such that V ∩ {x} = ∅. Thus y ∈ (V ∪ U) but not x and x ∩ U is a ngb-D-set. Hence x is ngb-D1.

Remark 3.14: It is clear that an ngb-T0 topological space (X, τ) is not a ngb-D1 iff there is a ngb-neat point in X. It is unique because x and y both are ngb-neat point in X, then at least one of them say x has an ngb-neighbourhood U containing x but not y. This is a contradiction since U \ X.

Definition 3.15: A topological space (X, τ) is symmetric if for x and y in X, x ∈ ngb-cl ({y}) ⇒ y ∈ ngb-cl ({x}).

Theorem 3.16: X is ngb-symmetric iff {x} is ngb-closed for x ∈ EX.

Proof: Assume that x ∈ ngb-cl ({y}) but y \ ngb-cl ({x}). This implies (ngb-cl ({x}))^c contains y. Hence the set {y} is a subset of (ngb-cl ({x}))^c. This implies ngb-cl ({y}) is a subset of (ngb-cl ({x}))^c. Now (ngb-cl ({x}))^c contains x which is a contradiction. Conversely, Suppose that {x} ⊆ E\ ngbBO(X, τ) but ngb-cl ({y}) which is a subset of E^c and x \ E. But this is a contradiction.

Theorem 3.17: A topological space (X, τ) is a ngb-T1 iff the singletons are ngb-closed sets.

Proof: Let (X, τ) be a ngb-T1, and x be any point of X. Suppose y ∈ {x}^c. Then x ≠ y and so there exists a ngb-open set U such that y ∈ U \ x \ U. Consequently, y ∈ U ∪ y ∈ (x)^c which is ngb-open. Conversely suppose {x} is ngb-closed for every x ∈ X. Let x, y ∈ X with x ≠ y. Then x ≠ y ⇒ y ∈ (x)^c. Hence (x)^c is a ngb-open set containing y but not x. Similarly (y)^c is a ngb-open set containing x but not y. Hence X is a ngb-T1-space.

Corollary 3.18: If X is a ngb-T1, then it is ngb-symmetric.

Proof: In a ngb-T1 space, singleton sets are ngb-closed. By theorem 3.17, and by theorem 3.16, the space is ngb-symmetric.

Corollary 3.19: The following statements are equivalent
(i) X is ngb-symmetric and ngb-T0
(ii) X is ngb-T1.

Proof: By corollary 3.18 and remark 3.7 it suffices to prove (1) ⇒ (2). Let x ≠ y and by ngb-T0 assume that x ∈ E, c \ (y)^c for some G_1 \ ngbBO(X, τ). Then x \ ngb-cl ({y}) and hence y \ ngb-cl ({x}). There exists G_1 \ ngbBO(X, τ) such that y ∈ G_1 ∩ (X)^c. Hence (x, τ) is a ngb-T1 space.

Theorem 3.15: For a ngb-symmetric topological space (X, τ), the following are equivalent.
1. X is ngb-T0
2. X is ngb-D1
3. X is ngb-T1.


4. Applications

Theorem 4.1: If f : (X, τ) → (Y, σ) is a ngb-continuous surjective function and S is a D-set in (Y, σ), then the inverse image of S is a ngb-D-set of (X, τ).

Proof: Let U_1 and U_2 be two open sets of (Y, σ). Let S = U_1 ∩ U_2 be a D-set and U_1 ≠ Y. We have f^(-1)(U_1) ∈ ngbBO(X, τ) and f^(-1)(U_2) ∈ ngbBO(X, τ) and f^(-1)(U_1) ≠ X. Hence f^(-1)(S) = f^(-1)(U_1 ∩ U_2) = f^(-1)(U_1) ∩ f^(-1)(U_2). Hence f^(-1)(S) is a ngb-D-set.

Theorem 4.2: If f : (X, τ) → (Y, σ) is a ngb-irresolute surjection and E is a ngb-D-set in Y, then the inverse image of E is an ngb-D-set in X.

Proof: Let E be a ngb-D-set in Y. Then there are ngb-open sets U_1 and U_2 in Y. Since E = U_1 ∩ U_2 and U_1 ≠ Y and f is ngb-irresolute, f^(-1)(U_1) and f^(-1)(U_2) ngb-open in X. Since U_1 ≠ Y, we have f^(-1)(U_1) ≠ X. Hence f^(-1)(E) = f^(-1)(U_1) ∩ f^(-1)(U_2) = f^(-1)(U_2) ∩ f^(-1)(U_1) is a ngb-D-set.

Theorem 4.3: If (Y, σ) is a D1 space and f : (X, τ) → (Y, σ) is a ngb-bijective function, then (X, τ) is a ngb-D1 space.

Proof: Suppose Y is a D1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is a D1 space, then there exists D-sets S_1 and S_2 of Y containing f(x) and f(y) respectively such that f(x) ∉ S_1 and f(y) ∉ S_2. By theorem 4.1 f^(-1)(S_1) and f^(-1)(S_2) are ngb-D- sets in X containing x and y respectively such that x ∉ f^(-1)(S_1) and y ∉ f^(-1)(S_2). Hence X is a ngb-D1 space.
Theorem 4.4: If Y is πgbd-f and f: (X,τ)→ (Y,σ) is πgbd-irresolute and bijective, then (X,τ) is πgbd-D1.

Proof: Suppose Y is πgbd-D1 and f is bijective, πgbd-irresolute. Let x,y be any pair of distinct points in X. Since f is injective and Y is πgbd-D1, there exists πgbd-D1 sets Gx and Gy in Y containing f(x) and f(y) respectively such that f(x) ∈ Gy and f(y) ∈ Gx. By theorem 4.2, f1(Gx) and f1(Gy) are πgbd-D1 sets in X containing x and y respectively. Hence X is πgbd-D1.

Theorem 4.5: A topological space (X, τ) is a πgbd-D1 if for each pair of distinct points x,y ∈ X, there exists a πgbd-irresolute surjective function f: (X,τ)→ (Y,σ), where (Y,σ) is a D1 space such that f(x) and f(y) are distinct.

Proof: Let x and y be any pair of distinct points in X. By hypothesis, there exists a πgbd-continuous surjective function f of a space (X, τ) onto a D1-space (Y, σ), such that f(x)≠f(y). Hence there exists disjoint D1 sets Sx and Sy in Y such that f(x)∈Sx and f(y)∈Sy. Since f is πgbd-irresolute and surjective, by theorem 4.1, f1(Sx) and f1(Sy) are disjoint πgbd-D1 sets in X containing x and y respectively. Therefore, (X, τ) is a πgbd-D1 set.

Theorem 4.6: X is πgbd-D1 if for each pair of distinct points x,y ∈ X, there exists a πgbd-irresolute surjective function f: (X,τ)→ (Y,σ), where Y is πgbd-D1 space such that f(x) and f(y) are distinct.

Proof: Necessity: For every pair of distinct points x,y ∈ X, if x≠y, then πgbd-irresolute function f exists such that f(x)≠f(y).

Sufficiency: Let x≠y

πgbd-irresolute surjection and (X,τ) is πgbd-compact. Hence Y is πgbd-set. Therefore Y is πgbd-D1.

Definition 4.7: A topological space (X, τ) is said to be πgbd-D-connected if every pair of distinct points x,y ∈ X is such that (X, τ) cannot be expressed as the union of two disjoint non-empty πgbd-D-sets.

Theorem 4.8: If (X, τ) → (Y, σ) is πgbd-continuous surjection and (X, τ) is πgbd-D-connected, then (Y, σ) is D-connected.

Proof: Suppose Y is not D-connected. Let Y=A∪B where A and B are two disjoint non empty D-sets in Y. Since f is πgbd-continuous and onto, X=f1(A)∪f1(B) where f1(A) and f1(B) are disjoint non-empty πgbd-D-sets in X. This contradicts the fact that X is πgbd-D-connected. Hence Y is D-connected.

Theorem 4.9: If (X, τ) → (Y, σ) is πgbd-irresolute surjection and (X, τ) is πgbd-D-connected, then (Y, σ) is πgbd-D-connected.

Proof: Suppose Y is not πgbd-D-connected. Let Y=A∪B where A and B are two disjoint non empty πgbd-D-sets in Y. Since f is πgbd-irresolute and onto, X=f1(A)∪f1(B) where f1(A) and f1(B) are disjoint non-empty πgbd-D-sets in X. This contradicts the fact that X is πgbd-D-connected. Hence Y is πgbd-D-connected.

Definition 4.10: A topological space (X, τ) is said to be πgbd- compact if every cover of X by πgbd-D-sets has a finite subcover.

Theorem 4.11: If a function f: (X, τ)→ (Y, σ) is πgbd-continuous surjection and (X, τ) is πgbd-compact then (Y, σ) is D-compact.

Proof: Let f: (X, τ)→ (Y, σ) is πgbd-continuous surjection. Let A={x∈A} be a cover of Y by D-set. Then {f1(A),f1(B),…,f1(A)} is a cover of X by πgbd-D-set. Since X is πgbd-compact, every cover of X by πgbd-D-set has a finite subcover, say {f1(A),f1(B),…,f1(A)}. Since f is onto, {A1,A2,…,An} is a cover of Y by D-set has a finite subcover. Therefore Y is πgbd-compact.

Theorem 4.12: If a function f: (X, τ)→ (Y, σ) is πgbd-irresolute surjection and (X, τ) is πgbd D-compact then (Y, σ) is πgbd-compact.

Proof: Let f: (X, τ)→ (Y, σ) is πgbd-irresolute surjection. Let A={x∈A} be a cover of Y by πgbd-D-set. Then Y=∪A, then X=f1(Y)=f1(∪Ai)=∪f1(Ai). Since f is πgbd-irresolute, for each i∈A, f1(Ai) is a cover of X by πgbd-set. Since X is πgbd-compact, every cover of X by πgbd-D-set has a finite subcover, say {f1(A1),f1(A2),…,f1(An)}. Since f is onto, {A1,A2,…,An} is a cover of Y by πgbd-D-set has a finite subcover. Therefore Y is πgbd-compact.

5. πgbd-R0 spaces and πgbd-R1 spaces

Definition 5.1: Let (X, τ) be a topological space then the πgbd-closure of A denoted by πgbd-cl(A) is defined by πgbd-cl(A) = ∩ { F | F ∈ πGBO(X, τ) and F⊇A}.

Definition 5.2: Let x be a point of topological space X. Then x ∈ πgbd-cl(A) if and only if x ∈ U where πgbd-cl(U) and x∈U.

Definition 5.3: Let F be a subset of a topological space X. Then πgbd-Cl(Ker πgbd(F)) is defined and denoted by Ker πgbd(F) = { x ∈ X | x ∈ πgbd-cl(U) and F⊂U }.

Lemma 5.4: Let (X,τ) be a topological space and x ∈ X. Then Ker πgbd(A) = { x ∈ X | πgbd-cl(A)∩A ≠∅ }.

Proof: Let x ∈ Ker πgbd(A) and πgbd-cl(A)∩A = F. Hence x ∈ X−πgbd-cl(A) which is an πgbd-open set containing A. This is impossible, since x ∈ Ker πgbd(A).

Consequently, πgbd-cl(A)∩A = F Let πgbd-cl(A)∩A = F and x ∈ Ker πgbd(A). Then there exists an πgbd-open set G containing A and x ∈ G. Let y ∈ πgbd-cl(A)∩A. Hence G is an πgbd- neighbourhood of y where x ∈ G. By this contradiction, x ∈ Ker πgbd(A).

Definition 5.5: Let (X,τ) be a topological space and x ∈ X. Then y ∈ Ker πgbd(A) if and only if x ∈ πgbd-cl(A).
Theorem 5.6: The following statements are equivalent for any two points x and y in a topological space (X, τ): (1) Ker(π)(x) ≠ Ker(π)(y); (2) πgbc-cl{x} ≠ πgbc-cl{y}.

Proof: (1) ⇒ (2): Suppose that Ker(π)(x) ≠ Ker(π)(y) then there exists a point z in X such that z ∈ Ker(π)(x) and z ∈ Ker(π)(y). It follows from z ∈ Ker(π)(y) that {x} ∩ πgbc-cl{z} ≠ ∅. This implies that x ∈ πgbc-cl{z}). By z ∈ Ker(π)(y), we have {y} ∩ πgbc-cl{z} = ∅. Since x ∈ πgbc-cl{z} , πgbc-cl{x} ⊂ πgbc-cl{y} and {y} ∩ πgbc-cl{z} = ∅. Therefore, πgbc-cl{x} ≠ πgbc-cl{y}. Now Ker(π)(x) ≠ Ker(π)(y) implies that πgbc-cl{x} ≠ πgbc-cl{y}.

(2) ⇒ (1): Suppose that πgbc-cl{x} ≠ πgbc-cl{y}. Then there exists a point z in X such that z ∈ πgbc-cl{x} and z ∉ πgbc-cl{y} and therefore Ker(π)(x) ≠ Ker(π)(y).

Definition 5.7: A topological space X is said to be πgbc-R0 iff πgbc-cl{x} ⊂ G whenever x ∈ G ∈ πGBO(X).

Definition 5.8: A topological space (X, τ) is said to be πgbc-R1 if for any x, y in X with πgbc-cl{x} ≠ πgbc-cl{y}), there exists disjoint πgbc-open sets U and V such that πgbc-cl{x} ⊂ U and πgbc-cl{y} ⊂ V.

Example 5.9: Let X = {a, b, c, d}, τ = {∅, {b}, {a, b}, {b, c}, {a, b, c}, X}. πGBO(X, τ) = P(X)

Theorem 5.10: X is πgbc-R0 iff given x ∈ Y ∈ πgbc-cl{y}.

Proof: Let X be πgbc-R0 and let x ∈ Y ∈ πgbc-cl{y}. Suppose U is a πgbc-open set containing x but not y, then y ∈ πgbc-cl{y} ⊂ X-U and hence x ∈ πgbc-cl{y}. Hence πgbc-cl{x} = πgbc-cl{y}.

Conversely, let x ∈ Y such that πgbc-cl{x} = πgbc-cl{y}. This implies πgbc-cl{x} ⊂ X-πgbc-cl{y} = U(say), a πgbc-open set in X. This is true for every πgbc-cl{y}. Thus ∩πgbc-cl{x} ⊂ U, where x ∈ πgbc-cl{x} ⊂ U ⊂ πGBO(X). This implies ∩πgbc-cl{x} ⊂ X-U, where x ∈ U ⊂ πGBO(X). Hence X is πgbc-R0.

Theorem 5.11: The following statements are equivalent
(i) X is πgbc-R0-space
(ii) For each x ∈ X, πgbc-cl{x} ⊂ Ker(π)(x)
(iii) For any πgbc-closed set F and a point x ∈ F, there exists U ∈ πGBO(X) such that x ∈ U and F ⊂ U.

(iv) Each πgbc-closed F can be expressed as F = ∩{G: G is πgbc-open and F ⊂ G}.
(v) Each πgbc-open G can be expressed as G = U {A: A is πgbc-closed and A ⊂ G}.
(vi) For each πgbc-closed set x ∈ F implies πgbc-cl{x} ∩ F = ∅.

Proof: (i) ⇒ (ii): For any x ∈ X, we have Ker(π)(x) = ∅ U {U ∈ πGBO(X)}. Since X is πgbc-R0, there exists an πgbc-open set containing x contains πgbc-cl{x} which implies Ker(π)(x) ∩ F = ∅.

(ii) ⇒ (iii): Let x ∈ πGBO(X). Then for any y ∈ F, πgbc-cl{y} ⊂ F and so x ∈ πgbc-cl{y} ⇒ x ∈ πgbc-cl{y}. That is there exists U ∈ πGBO(X) such that y ∈ U and x ∉ U. Let U = U ∪ {U ∈ πGBO(X) such that y ∉ U and x ∉ U}. Then U is πgbc-open such that x ∈ U and F ⊂ U.

(iii) ⇒ (iv): Let G be any πgbc-closed set and N = ∪ {G: G is πgbc-open and F ⊂ G}. Then FcN = ∅. Let x ∈ F, then by (iii) there exists G ∈ πGBO(X) such that x ∈ G and F ⊂ G, hence x ∈ N such that x ∈ N and Fc ⊂ N. From (i) and (2), each πgbc-closed F = ∩{G: G is πgbc-open and F ⊂ G}.

(iv) ⇒ (v): Obvious.

(v) ⇒ (vi): Let x ∈ πGBO(X). Then X-F = G is a πgbc-open set containing x. Then by (v), G can be expressed as the union of πgbc-closed sets A ∈ G and so there is an open GBC(X) such that x ∈ G and hence πgbc-cl{x} ⊂ G implies πgbc-cl{x} ∩ F = ∅.

(vi) ⇒ (i): Let x ∈ πGBO(X), then x ∈ GBC(X) which is πgbc-closed set. By (vi) πgbc-cl{x} ∩ (X-G) = ∅, implies πgbc-cl{x} ⊂ X. Thus X is πgbc-R0-space.

Theorem 5.12: A topological space (X, τ) is an πgbc-R0-space if and only if for any x and y in X, πgbc-cl{x} ≠ πgbc-cl{y} implies πgbc-cl{x} ∩ πgbc-cl{y} = ∅.

Proof: Necessity. Suppose that (X, τ) is πgbc-R0 and x, y ∈ X such that πgbc-cl{x} ≠ πgbc-cl{y}. Then, there exist z ∈ πgbc-cl{x} such that z ∈ πgbc-cl{y} (or z ∈ πgbc-cl{y}) such that z ∉ πgbc-cl{x}. There exists V ∈ πGBO(X) such that y ∉ V and z ∈ V. Hence x ∈ V. Therefore, we have x ∈ πgbc-cl{y}.

Thus x ∈ (πgbc-cl{y}) ∩ GBC(X), which implies πgbc-cl{x} ⊂ (πgbc-cl{y}) and πgbc-cl{x} ∩ πgbc-cl{y} = ∅.

Sufficiency. Let V ∈ πGBO(X) and let x ∈ V. To show that πgbc-cl{x} ⊂ V. Let y ∈ V, i.e., y ∈ V. Then x ∉ y and x ∈ πgbc-cl{y} (this is false). The same argument can be applied to πgbc-cl{y} and πgbc-cl{x} = ∅. Hence πgbc-cl{x} and therefore πgbc-cl{y} ⊂ V.

Theorem 5.13: A topological space (X, τ) is an πgbc-R0 space if and only if for any x and y in X, Ker(π)(x) ≠ Ker(π)(y) implies Ker(π)(x) ∩ Ker(π)(y) = ∅.

Proof: Suppose that (X, τ) is an πgbc-R0 space. Thus by Lemma 5.6, for any points x and y in X if Ker(π)(x) ≠ Ker(π)(y) then πgbc-cl{x} ≠ πgbc-cl{y}. Now to prove that Ker(π)(x) ∩ Ker(π)(y) = ∅.
From (ii), it follows that $\pi_{gb}(x) \subseteq \pi_{gb}(y)$. By $z \in \text{Ker}_{\pi_{gb}}(x) \cap \text{Ker}_{\pi_{gb}}(y)$, it follows that $x \in \pi_{gb}(z)$. Since $x \in \pi_{gb}(z)$ and $y \in \pi_{gb}(z)$, it is a contradiction. Therefore, we have $\text{Ker}_{\pi_{gb}}(x) \subseteq \text{Ker}_{\pi_{gb}}(y)$. Conversely, let $(X, \tau)$ be a topological space such that for any points $x$ and $y$ in $X$, $\pi_{gb}(x) \subseteq \pi_{gb}(y)$ implies $x \in \text{cl}_{\pi_{gb}}(y)$. Then $z \in \pi_{gb}(x) \subseteq \pi_{gb}(y)$ if and only if $y \in \pi_{gb}(z)$. Since $x \in \pi_{gb}(z)$, hence $\pi_{gb}(x) \subseteq \pi_{gb}(y)$. This is a contradiction. Conversely, let $y \in \pi_{gb}(z)$, then $\pi_{gb}(x) \subseteq \pi_{gb}(y)$. Therefore, $\text{cl}_{\pi_{gb}}(y) \subseteq \pi_{gb}(z)$.

**Theorem 5.14:** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is an $\pi_{gb}$-space;
2. $x \in \pi_{gb}(y)$ if and only if $y \in \pi_{gb}(x)$ for any points $x$ and $y$.

**Proof:** (1) $\Rightarrow$ (2): Assume that $x$ is an $\pi_{gb}$-space. Let $x \in \pi_{gb}(y)$ and $y \in G$ be any $\pi_{gb}$-open subset such that $y \in \pi_{gb}(X)$. By hypothesis, $x \in G$. Hence $y \in \pi_{gb}(X)$.

(2) $\Rightarrow$ (1): Let $U$ be an $\pi_{gb}$-open set and $x \in U$. If $y \in U$, then $x \in \pi_{gb}(y)$ and hence $y \in \pi_{gb}(x)$. This implies that $\pi_{gb}(x) \subseteq U$. Hence $(X, \tau)$ is an $\pi_{gb}$-space.

**Theorem 5.15:** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is an $\pi_{gb}$-space;
2. $\pi_{gb}(x) \subseteq \text{cl}_{\pi_{gb}}(x)$ for all $x \in X$.

**Proof:** (1) $\Rightarrow$ (2): Suppose that $(X, \tau)$ is an $\pi_{gb}$-space. By theorem 5.11, $\pi_{gb}(x) \subseteq \text{cl}_{\pi_{gb}}(x)$ for each $x \in X$. Let $y \in \pi_{gb}(x)$, then $x \in \pi_{gb}(y)$ and $\pi_{gb}(x) \subseteq \pi_{gb}(y)$. Therefore, $y \in \pi_{gb}(x)$ and hence $\pi_{gb}(x) \subseteq \pi_{gb}(x)$. This shows that $\pi_{gb}(x) \subseteq \text{cl}_{\pi_{gb}}(x)$.

(ii) $\Rightarrow$ (i): Obvious from 5.13E.

**Theorem 5.16:** For a topological space $(X, \tau)$, the following are equivalent:

1. $(X, \tau)$ is an $\pi_{gb}$-space;
2. $F \subseteq \pi_{gb}(x)$ for all $x \in X$.
3. $x \in \pi_{gb}(F)$.
4. $x \in \text{cl}_{\pi_{gb}}(F)$.

**Proof:** (i) $\Rightarrow$ (ii): Let $F$ be an $\pi_{gb}$-closed set and $x \in \pi_{gb}(F)$. Then $x \in \text{cl}_{\pi_{gb}}(F)$ and $x \in \text{cl}_{\pi_{gb}}(F)$.

References