Fuzzy game value of the Interval matrix

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ABSTRACT:

In this paper we present a method to identify the fuzzy game value of the matrix with interval data. The ideas of crisp matrix games are extended to fuzzy matrix games. This result is applied to a three player game.

Keyword: Crisp game value, Fuzzy game value, crisply related, fuzzily related and saddle value.

1. Introduction

Game theory has been used extensively to analyze conflict and co-operative situations in Economics, sociology, etc. One of the basic problems in the game theory is the two- player zero sum game.

A fundamental problem of classical game theory is that the player makes decisions with crisp data. However, in a real life world, most games always take place in uncertain environments. Because of uncertainty in real world applications, payoffs of a matrix game may not be a fixed number. This situation forces us to introduce fuzzy matrix game. In this paper, we consider the payoffs as interval numbers.

1.1 Crisp game value of the Matrix:

A game can be expressed as

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -4 & -3 \\ -1 & 5 & -1 \end{bmatrix} \quad (1)$$

In this game, the players X and Y have 3 and 4 strategies respectively. For player X, minimum value in each row represents the least gain (payoff) to him if he chooses his particular strategy. He will then select the strategy that maximizes his minimum gain.

Similarly, for player Y, the maximum value in each column represents the maximum loss to him if he chooses his particular strategy. He will then select the strategy that minimizes his maximum losses.

1.2 Definition:

If the maximum value equals the minimax value, then the game is said to have a *saddle value*(equilibrium point) and the corresponding strategies which give the saddle point are called optimal strategies.[1] and [6].The amount of payoff at an equilibrium point is called the crisp game value of the matrix.

In (1), a_{11} is the crisp game value of the matrix. That is $a_{11} = 1$.

1.3 Interval Data:

Due to uncertainty, the payoffs in a matrix are not fixed numbers. Hence fuzzy games have been studied. For further studies see [3], [5] and [7]. To model such uncertainty, we use interval valued matrix.

1.4 Definition:

Let A= $\lfloor a_{ij} \rfloor$ be an mxn rectangular interval valued matrix. This matrix defines a zero sum interval matrix game provided whenever, the row player X uses his ith strategy and the column player Y uses his jth strategy. Then X wins and Y loses a common value x $\in \lfloor a_{ij} \rfloor$.

1.5 Example:

$$A = \begin{bmatrix} [1,2] & [7,8] & [-3,-1] & [-5,-3] \\ [4,6] & [0,8] & [-1,3] & [4,5] \\ [-7,-3] & [-2,-1] & [1,1] & [4,5] \end{bmatrix}$$

In this game if X chooses row one and Y chooses column two, then X wins an amount x $\in [7, 8]$ and Y loses the same amount.

2. Interval Arithmetic

The extension of ordinary arithmetic to intervals is known as interval arithmetic.

Let $A = [a_L, a_R]$, $B = [b_L, b_R]$ be two intervals.

Then,

(i) $A + B = [a_L + b_L,\ a_R + b_R]$.

(ii) $A - B = [a_L - b_R,\ a_R - b_L]$.

(iii) AB $= [\min\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}, \max\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}]$

(iv) λ A= [$\lambda\, a_L$, $\lambda\, a_R$], if $\lambda \geq 0$ and = [$\lambda\, a_R$, $\lambda\, a_L$], if λ <0.

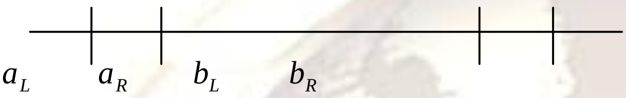
Similarly the other binary operations are defined

2.1Coimparison of Intervals

Comparison of two intervals is very important problem in interval analysis. In this section we consider the order relation (≤ and ≥) between intervals. Let a and b be two non-empty intervals.

2.2 Disjoint Intervals:

Case (i): If a= [a_L , a_R], b= [b_L , b_R] and if $a_R < b_L$ then a<b or b>a crisply, which is similar to the definition of comparisons used in [4]



2.3 Example:

a = [2, 3] and b= [5, 6] be two intervals. Here a < b.

2.4 Equal Intervals:

Case (ii): a=b iff a≤b and a≥b.

2.5: Overlapping Intervals:

If $a_L < b_L < a_R < b_R$ then for any x in [a_L , b_L], a<b.That is x ∈ a, is less than every payoff in b.

If x ∈ [b_L , a_R], then every x ∈ a, is less than or equal to b. Therefore a≤b (crisply).

Similarly if y ∈ [a_R , b_R], then y in b is greater than or equal to a. That is b≥a crisply.

3. Nested sub intervals

In terms of fuzzy membership

$$a \leq b = \frac{b_R - a_R}{b_R - b_L}$$

If b ⊆ a then

$$a \leq b = \frac{b_L - a_L}{a_R - a_L}$$

Consolidating the above discussions we define the fuzzy operations $\prec$ *and* $\succ$ as follows.

3.1 Definition:

The binary fuzzy operator $\prec$ of two intervals a and b returns a real number between 0 and 1 as follows The binary fuzzy operator $\prec$ of two intervals a and b returns a real number between 0 and 1 as follows

a $\prec$ b = 1, if a = b or $a_R < b_L$, a ≠ b; or $a_L < b_L < a_R < b_R$.

0, if $b_R < a_L$, $b_L < a_L < b_R < a_R$.

$\frac{b_L - a_L}{a_R - a_L}$, if $a_L < b_L < b_R < a_R$.

$\frac{b_R - a_R}{b_R - b_L}$, if $b_L < a_L < a_R < b_R$.

These values $\frac{b_L - a_L}{a_R - a_L}$ and $\frac{b_R - a_R}{b_R - b_L}$ moves from 0 to 1 when x moves a_L to b_L and a_R to b_R from left to right. Only when a=b, a $\prec$ b takes the value 1.Using simple algebraic operations, it can be seen that the membership value for b $\succ$ a =1- a $\prec$ b.

3.2 Definition:

The binary fuzzy operator ≥ of two intervals a and b is defined as

a ≥ b= 1 if a = b ;or $b_R < a_L$ or $b_L < a_L < b_R < a_R$.

0 if $a_R < b_L$ a ≠ b, $a_L < b_L < a_R < b_R$

$\frac{a_R - b_R}{a_R - a_L}$ if $a_L < b_L < b_R < a_R$.

$\frac{a_L - b_L}{b_R - b_L}$ if $b_L < a_L < a_R < b_R$.

The relations of two intervals can now be either crisp of fuzzy as described below.

3.3 Definition:

If the values of a ≤ b is exactly one or zero then we say that a and b are crisply related otherwise we say that they are fuzzily related.

4. Crisp Game value of the Matrix:

The ideas and concept of crisp game value if the matrixes can be extended to the matrix games with interval data where entries are crisply related.

4.1 Definition:

Let A be an mxn interval game matrix such that all the intervals in the same row (or column) of A are crisply related. If there exists a $g_{ij} \in A \ni a_{ij}$ is simultaneously crisply less than or equal to a_{ik} $\forall k = 1.........n$ and crisply ≥

$g_{lj} \forall l = 1...m$, then the interval g_{ij} is called a saddle interval of the game. The value of the saddle interval is called the crisp game value of interval matrix game.

By the definition above, to determine whether an interval game matrix has a crisp game value, one needs to verify the following:-

(i) For each row($1 \le j \le$m) find the entry a_{ij}, that is crisply less than or equal to all other entries of the ith row.

(ii) For each column ($1 \le j \le$m) find the entry a_{ij} that is crisply greater than or equal to all other entries of the j th column.

(iii) Determine if there is an entry a_{ij} ,that is simultaneously the minimum of the i th row and the maximum of the jth column.

(iv) If any of the above values cannot be determined for the matrix game, the crisp game value cannot be determined.

4.2 Example:

Y

A= X

$$\begin{bmatrix} [1,2] & [7,8] & [-3,-1] & [-5,-3] \\ [4,6] & [0,8] & [-1,3] & [4,5] \\ [-7,-3] & [-2,-1] & [1,1] & [4,5] \end{bmatrix}$$

We see that in the above game matrix a_{14}, a_{23}, a_{31} are the minimum of rows 1, 2 and3 respectively and a_{21}, a_{12}, a_{23}, a_{34} are the maximum of column 1,2,3, and 4 .Therefore a_{23} is the saddle interval of the matrix A. The value of this interval is the crisp game value of the interval matrix game. The corresponding strategies which give the saddle interval values are the optimal strategies of the crisp game value. That is the optimal strategies for both X and Y in a crisp interval valued matrix game are defined as

(a) X should select any row containing a saddle interval and

(b) Y should select any column containing a saddle interval.

4.3 Definition:

The game value of a crisply interval valued matrix game is its saddle interval. This interval game is fair, if its saddle interval is symmetric with respect to zero. That is in the form of [-a, a], a≥0.

In the above example, the saddle interval is a_{23}= [0, 1].Here 0.5 is the midpoint. Hence the game is unfair since the row player has an advantage of 0.5.

5. Fuzzy game value of the interval matrix

The crisp relativity may not be satisfied for all intervals in the same row (or column) .We now define the fuzzy memberships of an interval being a minimum and a maximum of an interval vector R and then we define the notion of a least and greatest interval in R.

5.1 Definition:

The least interval of the vector R is defined as max {min { $r_i \le r_j$ }} =l

$1 \le i \le n, 1 \le j \le m$

and similarly the greatest interval of the vector R is defined as min {max { $r_i \ge r_j$ }} =g

$1 \le j \le m, 1 \le i \le n$

5.2 Example:

Find the least and the greatest interval of the vector R= {[1, 5], [2, 8], [3, 5]}

(i) [1, 5] ≤ [2, 8] =1
[1, 5] ≤ [3, 5] = 1

Therefore min [1, 5] =1.

(ii) [2, 8] ≤ [1, 5] =0
[2, 8] ≤ [3, 5] = $\frac{1}{6}$

Therefore min [2, 8] =0.

(iii) [3, 5] ≤ [1, 5] =0
[3, 5] ≤ [2, 8] = $\frac{1}{2}$

Therefore min [3, 5] = 0.
Therefore max {1, 0, 0} = 1.
Hence [1, 5] is the least interval.

(i)[1, 5] ≥ [2, 8] =0
[1, 5] ≥ [3, 5] =0
Therefore max [1, 5] =0.
(ii) [2, 8] ≥ [1, 5] =1
[2, 8] ≥ [3, 5] = $\frac{1}{2}$.
Therefore max [2, 8] =1.

(iii) [3, 5] ≥ [1, 5] = $\frac{1}{2}$
[3, 5] ≥ [2, 8] = $\frac{1}{6}$.

Therefore max [3, 5] = $\frac{1}{2}$.

Hence min {0, 1, $\frac{1}{2}$ }= 0.

This corresponds to [1, 5] which is the largest interval.

Hence we define fuzzy game value of interval matrix as follows.

5.3 Definition:

Let R be an mxn interval game matrix. If there is an $r_{ij} \in R \ni r_{ij}$ is simultaneously the least and the greatest interval for the ith row and jth column of R , then that interval value is the fuzzy game value of the interval valued matrix game.

6. Three player game

The above result can be extended to three player game. This will be a special case where the players A and B had the choice of choosing the strategies X1 and X2, whereas the player C will have a fixed strategy.

For example if the matrix game is given as follows:

C
B

		X1	X2
A	X1	$[a_1, b_1]$	$[a_2, b_2]$
	X2	$[a_3, b_3]$	$[a_4, b_4]$

Here the players A and b have strategies X1 and X2.We assume that in the first case the player C chooses X1 and in the second case he chooses X2.

Case (i):

When the player C chooses X1, if the payoff matrix is given as:

C
B

		X1	X2
A	X1	$[a_1, b_1]$	$[a_2, b_2]$
	X2	$[a_3, b_3]$	$[a_4, b_4]$

Find the saddle interval of the vectors $\{[a_1,b_1], [a_2,b_2], [a_3,b_3], [a_4,b_4]\}$ if it exists. If the saddle interval falls into the column corresponding to the strategy X1, then the player C has a choice of winning the game with the fuzzy membership corresponding to that interval. Otherwise, he looses the game.

Case (ii):

Similarly, when the third player C chooses X2, if the payoff matrix is given as follows:

C
B

		X1	X2
A	X1	$[c_1, d_1]$	$[c_2, d_2]$
	X2	$[c_3, d_3]$	$[c_4, d_4]$

Then as in the above case find the saddle interval of the vectors $\{[c_1,d_1],[c_2,d_2],[c_3,d_3],[c_4,d_4]\}$. If the saddle interval falls into the column of X2, then the player C has a choice of winning the game with the fuzzy membership corresponding to the interval. Otherwise he loses the game. This result can be extended to many strategies X1, X2, X3 …Xn provided each time the payoff matrices are given when the third player C chooses one of the strategies. This is illustrated in the following example.

6.1 Example:

Case (i):

When C chooses X1, if the payoff matrix is given as below, then

C
B

		X1	X2
A	X1	[1,6]	[7,8]
	X2	[2,6]	[3,4]

We find the saddle interval of {[1, 6], [7, 8], [2, 6], [3, 4]}

Least interval:

I (a) $[1, 6] \prec [7, 8] = 1$

(b) $[1, 6] \prec [2, 6] = \frac{1}{5}$

(c) $[1, 6] \prec [3, 4] = \frac{2}{5}$

Therefore $\min \{1, \frac{1}{5}, \frac{2}{5}\} = \frac{1}{5}$

II (a) $[7, 8] \prec [1, 6] = 0$

(b) $[7, 8] \prec [2, 6] = 0$

(c) $[7, 8] \prec [3, 4] = 0$

Therefore min {0, 0, 0} =0

III (a) $[2, 6] \prec [1, 6] = 0$

(b) $[2, 6] \prec [7, 8] = 1$

(c) $[2, 6] \prec [3, 4] = \frac{1}{4}$

Therefore min $\{0, 1, \frac{1}{4}\}=0$

IV (a) $[3, 4] \prec [1, 6] = \frac{2}{5}$

(b) $[3, 4] \prec [2, 6] = \frac{1}{2}$

(c) $[3, 4] \prec [7, 8] = 1$

Therefore min $\{\frac{2}{5}, \frac{1}{2}, 1\}=\frac{2}{5}$.

Hence max $\{\frac{1}{5}, 0, 0, \frac{2}{5}\}=\frac{2}{5}$

This corresponds to the interval $[3, 4] \prec [1, 6]$.
Therefore the least interval is [3, 4].

Greatest interval:

I (a) $[1, 6] \succ [2, 6]=0$

(b) $[1, 6] \succ [7, 8] = 0$

(c) $[1, 6] \succ [3, 4] = \frac{2}{5}$

Therefore max $\{0, 0, \frac{2}{5}\}=\frac{2}{5}$.

II (a) $[2, 6] \succ [1, 6] = \frac{1}{5}$

(b) $[2, 6] \succ [7, 8] = 0$

(c) $[2, 6] \succ [3, 4] = \frac{1}{2}$

Therefore max $\{0, \frac{1}{5}, \frac{1}{2}\}=\frac{1}{2}$

III (a) $[7, 8] \succ [1, 6] = 1$

(b) $[7, 8] \succ [2, 6] = 1$

(c) $[7, 8] \succ [3, 4] = 1$

Therefore max {1, 1, 1} =1

IV (a) $[3, 4] \succ [1, 6]= \frac{2}{5}$

(b) $[3, 4] \succ [2, 6] = \frac{1}{4}$

(c) $[3, 4] \succ [7, 8] = \frac{2}{5}$.

Therefore max $\{\frac{2}{5}, \frac{1}{4}, \frac{2}{5}\}=\frac{2}{5}$.

Hence min $\{\frac{2}{5}, \frac{1}{2}, 1, \frac{2}{5}\}=\frac{2}{5}$

This corresponds to the interval [3, 4].and [1, 6]
Since [3, 4] is the least and the greatest interval, it is the saddle interval whose membership is 0.4.
If the third player C chooses X2, he wins and the game value lies in the interval [3, 4] with the fuzzy membership 0.4.If he chooses X1, he loses the game.

Case (ii):
Similarly when C chooses the strategy X2 if the payoff matrix is given as below, then in the same manner we find the saddle interval. If it falls in the second column the player C wins with the fuzzy membership of that interval. Otherwise he loses the game.

C

B

		X1	X2
A	X1	[3,12]	[7,10]
	X2	[5,10]	[4,5]

We find the saddle interval of the vectors {[3,12], [7,10],[5,10],[4,5]}

I (a) $[3, 12] \prec [7, 10] = \frac{4}{9}$

(b) $[3, 12] \prec [5, 10] = \frac{2}{9}$

(c) $[3, 12] \prec [4, 5] = \frac{1}{9}$

Min $\{\frac{4}{9}, \frac{2}{9}, \frac{1}{9}\}=\frac{1}{9}$.

II (a) $[7, 10] \prec [3, 12]=\frac{2}{9}$

(b) $[7, 10] \prec [4, 5]=0$

(c) $[7, 10] \prec [5, 10] =0$

Min $\{\frac{2}{9}, 0, 0\} =0$

III (a) $[5, 10] \prec [3, 12] = \frac{2}{9}$

(b) $[5, 10] \prec [7, 10] = \frac{2}{5}$

(c) $[5, 10] \prec [4, 5] = 0$

$$\text{Min}\ \{\frac{2}{9}, \frac{2}{5}, 0\} = 0$$

IV (a) $[4, 5] \prec [3, 12] = \frac{7}{9}$

(b) $[4, 5] \prec [7, 10] = 1$

(c) $[4, 5] \prec [5, 10] = 1$

$$\text{Min}\ \{\frac{7}{9}, 1, 1\} = \frac{7}{9}$$

$$\text{Max}\ \{0, \frac{1}{9}, 0, \frac{7}{9}\} = \frac{7}{9}$$

This corresponds to the interval [4, 5].

7.Conclusion

In this paper we have introduced two person zero-sum interval valued matrix games. The fuzzy game value of rectangular matrices can be obtained by verifying whether the given matrix is fuzzily related or not.

In some cases, during the comparison of intervals will satisfy both $\prec$ *and* $\succ$. But depending upon the membership values we can draw conclusions. The concept is extended to a three player game. This can also be extended to a multi-player game.

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