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Three Steps Block Predictor-Block Corrector Method For The Solution Of General Second Order Ordinary Differential Equations

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Abstract:

We consider three steps numerical integrator which is derived by collocation and interpolation of power series approximation solution to generate a linear multistep method which serves as the corrector. The corrector is then implemented in block method. We then propose another numerical integrator which is implemented in block method to serve as a constant order predictor to the corrector. Basic properties of the corrector viz, order, zero stability and stability interval are investigated. The new method was tested on some numerical examples and was found to give better approximation than the existing methods.

Keyword: Collocation, Interpolation, approximate solution, block method, order, zero stability.
AMS Subject classification: 65L05, 65L06, 65D30.

1 Introduction

This paper considers method of solving general second order initial value problems of ordinary differential equation of the form

$$y'' = f(x, y, y'),\ y^k(x_0) = y_0^k, k = 0,1 \qquad (1)$$

Where f is continuous within the interval of integration.

Method of reduction of higher order ordinary differential equation to systems of first order ordinary differential equation has been discussed by many scholars. These authors suggested that the method increased the dimension of the resulting system of the first order; hence it wastes computer and human effort. Among such authors are [5, 9, 11]. Scholars later adopted predictor–corrector method to solve higher order ordinary differential equation directly. These authors proposed implicit linear multistep methods which serve as the corrector, the predictors proposed to implement this corrector are in reducing order of accuracy. These methods are developed by adopting different approximate solution such as Power series, Chebychev polynomial, Newton polynomial, Lagrange polynomial, Fourier series to mention few. The major disadvantage of this method is that the predictors are reducing order predictors hence it has a lot of effect on the accuracy of the method. Among authors that proposed predictor corrector method are [6, 7, 12]. Other setbacks of this method are extensively discussed by [7].

In order to cater for the setbacks of the predictor-corrector method, scholars adopted block method for direct solution of higher order ordinary differential equation. This method has the properties of being self starting and gives evaluation at selected grid points without over lapping. It does not require developing predictors or starting values moreover it evaluates fewer functions per step. The major setback of this method is that the interpolation points cannot exceed the order of the differential equation, hence this method does not exhaust all interpolation points therefore methods of lower order are developed. Among these authors are [1, 7, 9, 10, 15, 13, 14, 15, 16]. Scholars later adopted block predictor-block corrector method to cater for some of the setbacks of block method. They proposed a method which is capable of exhausting all the possible interpolation point as the corrector and then proposed a constant order predictor using block method. The method forms a bridge between the predictor-corrector method and the block method. Among these authors are [2, 3, 4]. The major setback of this method is that the evaluation at selected points are overlapping hence the behaviour of the dynamical system at selected grid points cannot be mentioned.

In this paper, we proposed a method in which the corrector is implemented in block method; hence this method gives evaluation at selected grid points without overlapping.

2 Methodology

2.1 Development of the block corrector|

We consider a power series approximate solution of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \qquad (2)$$

Where r and s are the number of collocation and interpolation points respectively.

The second derivative of (2) gives

$$y'' = \sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2}$$

(3)

Substituting (3) into (1) gives

$$f(x,y,y') = \sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2}$$

(4)

Collocating (3) at x_{n+r}, $r = 0(1)3$ and interpolating (2) at x_{n+s}, $s = 0(1)2$ gives a system of non linear equation of the form

$$AU = B$$

(5)

$$U = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 0 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^{54} & x_{n+1}^6 \\ 0 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 \end{bmatrix}$$

$$, A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}, B = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}$$

Solving (5) for a_j's, using Gaussian elimination method and substituting back into the approximate solution, gives a continuous linear multistep method.

$$y(x) = \sum_{j=0}^{2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{3} \beta_j(x) f_{n+j} \tag{6}$$

Where the coefficient of y_{n+j} and f_{n+j} gives

$$\alpha_0 = -\frac{1}{6}\left(2t^6 - 18t^5 + 55t^4 - 60t^3 + 27t - 6\right)$$

$$\alpha_1 = \frac{1}{3}\left(2t^6 - 18t^5 + 55t^4 - 60t^3 + 24t\right)$$

$$\alpha_2 = -\frac{1}{6}\left(2t^6 - 18t^5 + 55t^4 - 60t^3 + 21t\right)$$

$$\beta_0 = \frac{1}{360}\left(10t^6 - 93t^5 + 305t^4 - 410t^3 + 180t^2 + 8t\right)$$

$$\beta_1 = \frac{1}{360}\left(100t^6 - 89t^5 + 2675t^4 - 2820t^3 + 936t\right)$$

$$\beta_2 = \frac{1}{360}\left(10t^6 - 99t^5 + 335t^4 - 390t^3 + 144t\right)$$

$$\beta_3 = \frac{1}{360}\left(3t^5 - 15t^4 + 20t^3 - 8t\right)$$

$$t = \frac{x - x_n}{h},\ y_{n+j} = y(x_n + jh),$$

$$f_{n+j} = f(x_n, y(x_n + jh), y'(x_n + jh))$$

Evaluating (6) at $t = 4$ gives

$$y_{n+3} = 3y_{n+2} + 3y_{n+1} + y_n + \frac{h^2}{12}\left(f_{n+3} + 9f_{n+2} - 9f_{n+1} - f_n\right) \tag{7}$$

Evaluating the first derivative of (6) at $t = 0,1$ gives

$$hy'_n = \frac{-2}{3}y_{n+2} + 8y_{n+1} - \frac{9}{2}y_n + \frac{h^2}{45}\left(f_n + 117f_{n+1} + 18f_{n+2} - f_{n+3}\right) \tag{8}$$

$$hy'_{n+1} = \frac{-17}{6}y_{n+2} - 14y_{n+1} + \frac{11}{6}y_n + \frac{h^2}{360}\left(47f_n + 679f_{n+1} - 121f_{n+2} + 7f_{n+3}\right) \tag{9}$$

Writing (7)-(9) in block form and solving for the independent solution at selected grid points gives

$$A^{(0)}Y_m = ey_n + cy'_n + df(y_n) + bF(Y_m)$$

(10)

Where
$$Y_m = \left[y_{n+1}, y_{n+2}, y_{n+3}\right]^T$$
$$y_n = \left[y_{n-1}, y_{n-2}, y_n\right]^T$$
$$y'_n = \left[y_{n-1}, y'_n, y'_{n+1}\right]^T$$
$$f(y_n) = \left[f_{n-1}, f_{n-2}, f_n\right]^T$$
$$F(Y_m) = \left[f_{n+1}, f_{n+2}, f_{n+3}\right]^T$$

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad e = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$c = \begin{bmatrix} 0 & \frac{17}{38} & \frac{21}{38} \\ 0 & \frac{14}{19} & \frac{24}{19} \\ 0 & \frac{33}{38} & \frac{81}{38} \end{bmatrix},$$

$$d = \begin{bmatrix} 0 & 0 & \frac{851}{13680} \\ 0 & 0 & \frac{127}{855} \\ 0 & 0 & \frac{267}{1520} \end{bmatrix},$$

$$b = \begin{bmatrix} \frac{-1653}{13680} & \frac{93}{13680} & \frac{-11}{13680} \\ \frac{399}{855} & \frac{111}{855} & \frac{-7}{855} \\ \frac{1539}{1520} & \frac{1701}{1520} & \frac{93}{1520} \end{bmatrix}$$

2.2 Development of Block predictor

[7] developed 3 steps block method of the form (10) which is writing explicitly to give

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{360}\left(97f_n + 114f_{n+1} - 39f_{n+2} + 8f_{n+3}\right)$$

$$y_{n+2} = y_n + 2hy_n' + \frac{h^2}{45}\left(28f_n + 66f_{n+1} - 6f_{n+2} + 2f_{n+3}\right)$$

$$y_{n+3} = y_n + 3hy_n' + \frac{h^2}{40}\left(39f_n + 108f_{n+1} + 27f_{n+2} + 6f_{n+3}\right)$$

$$y_{n+1}' = y_n' + \frac{h}{24}\left(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}\right)$$

$$y_{n+2}' = y_n' + \frac{h}{3}\left(f_n + 4f_{n+1} + f_{n+2}\right)$$

$$y_{n+3}' = y_n' + \frac{h}{8}\left(3f_n + 9f_{n+1} + 9f_{n+2} + 3f_{n+3}\right)$$

For the derivation and analysis of the block predictor, readers are referred to [7].

2.3 Implementation of the Method

Given the general block formula proposed by [7] in the form

$$Y_m = ey_n + h^{\mu}df(y_n) + bh^{\mu}F(Y_m)$$

(11)

We then proposed a prediction equation of the form

$$Y_m = ey_n + \sum_{\lambda=0}^{m} h^{\mu+\lambda}F^{\lambda}(y_n), \qquad (12)$$

Where $\mu = 2$ is the order of the differential equation. In this paper, $m = 2.$

$F^{\lambda}(y_n) = \frac{d}{dx^{\lambda}} f^{\lambda}(x, y, y')_{y_n}$ Substituting (10) into (11) gives

$$Y_m = ey_n + h^{\mu}dF(y_n) + bh^{\mu}F\left(ey_n + \sum_{\lambda=0}^{2} h^{\mu+\lambda} f^{\lambda} f(y)\right) \qquad (13)$$

Writing (10) as

$$Y_m^* = e^* y_n + chy_n' + d^* h^2 f(y_n) + bF(Y_m^*)$$

(14)

Substituting (13) in to (14) gives

$$Y_m^* = e^* y_n + chy_n' + d^* h^2 f(y_n) + bF(Y_m)$$

(15)

Equation (15) is our **block predictor-block corrector method**.

4.0 Analysis of basic properties of the method

4.1 Order of the method

We define a linear operator on the block (10) to give $\mathcal{L}\{y(x):h\} =$

$$y(x) - A^{(0)}Y_m - ey_n - chy_n' - df(y_n) - bF(Y_m) \qquad (16)$$

Expanding (16) in Taylor series gives

$\mathcal{L}\{y(x):h\} = c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + \cdots + c_p h^b(x)$

Definition: the block (10) and associated linear operator are said to have order p if $c_0 = c_1 = c_2 = \ldots = c_{p+1} = 0$ and $c_{p+2} \neq 0$. The term c_{p+2} is called the error constant and implies that the local truncation error is

$$t_{n+k} = c_{p+2}h^{(p+2)}y^{p+2}x_n + O(h^{p+3})$$

Hence the block (10) has order five with error constant $\left[\frac{97}{383040}, \frac{41}{23940}, \frac{9}{42560}\right]$

4.2 Zero Stability

A block method is said to be zero stable if as $h \to 0$, the root $r_j; j = 1(1)k$ of the first characteristic polynomial $\rho(R) = 0$ that is $\rho(R) = \det\left[\sum A^{(0)}R^{k-1}\right] = 0$ satisfying $|R| \leq 1$

and for those roots with $|R| \leq 1$ must have multiplicity equal to unity {see [9] for details}

Hence $\left| R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$

$R^3 - R^2 = 0$, $R^2(R-1) = 0$ $R = 0,0,1$

5.0 Numerical Experiments

We test our method with second order initial value problems.

Problem 1: Consider a non linear initial value problem

$$y'' - x(y') = 0, \quad y(0) = 1, \; y'(0) = \frac{1}{2}, \; h = 0.01$$

Exact solution $y(x) = 1 + \frac{1}{2}\log_e\left(\frac{2+x}{2-x}\right)$

This problem was solved by [1] where a block of order four was proposed for $h = \frac{0.1}{32}$ and [7] where a 3 steps block method was proposed for the same step size. The result is shown in table I

Problem II: We consider a highly oscillatory test problem

$y'' + \lambda^2 y = 0$, where $\lambda = 2$, with initial condition $y(0) = 1, y'(0) = 2, h = 0.01$

Exact solution; $y(x) = \cos 2x + \sin 2x$

We compare our result with [7] where a block method with step length of three was proposed

Table I for problem I

x	**Exact result**	**Computed result**	**Error**	**Error in [1]**
0.1	1.050041729278491	1.050041729278624	1.3300(-13)	4.9827(-11)
0.2	1.100335347710756	1.100335347732074	9.9498(-13)	4.1043(-10)
0.3	1.151140435936466	1.511404359399064	3.4396(-12)	1.4285(-09)
0.4	1.202732550540823	1.202732554062762	8.6803(-12)	3.5242(-09)
0.5	1.255412811882995	1.255412811901369	1.8373(-11)	7.2435(-09)
0.6	1.309519604203111	1.309519604338198	3.5087(-11)	1.3335(-08)
0.7	1.365443754271396	1.365443754334709	6.3313(-11)	2.2872(-08)
0.8	1.423645930193602	1.423648930303739	1.1013(-10)	3.7447(-08)
0.9	1.484700278594052	1.484700278782232	1.8818(-10)	5.9507(-08)
1.0	1.849306144334055	1.549306144656174	3.2211(-10)	9.2940(-08)

Table II for problem II

x	Exact result	Computed result	Error	Error in [7]
0.01	1.0197986733599109	1.0197986733589541	9.5379(-13)	-
0.02	1.0391894408476121	1.0391894408454274	2.1846(-12)	2.65(-06)
0.03	1.0581645464146487	1.0581645464109597	3.6890(-12)	3.98(-06)
0.04	1.0767164002717922	1.0767164002646123	7.1798(-12)	5.30(-06)
0.05	1.0948375819248539	1.0948375819138998	1.0965(-11)	6.62(-06)
0.06	1.1125208431427855	1.1125208431277691	1.5016(-11)	7.94(-06)
0.07	1.1297591108568736	1.1297591108357115	2.1162(-11)	9.25(-06)
0.08	1.1465454899898728	1.1465454899622722	2.7600(-11)	1.06(-06)
0.09	1.1628732662139456	1.1628732661796120	3.4333(-11)	1.19(-06)
0.10	1.1787359086363027	1.1787359085930647	4.3238(-11)	1.32(-06)

Conclusion:

We have proposed a 3 steps block predictor-block corrector method in this paper. This method has addressed some of the setbacks of block predictor corrector method proposed by [2, 3, 4]. The result has shown clearly that our method gives better approximation than the existing method.

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