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ON FUZZY TYCHONOFF THEOREM AND INITIAL AND FINAL TOPOLOGIES

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Abstract:

Much of topology can be done in a setting where open sets have fuzzy boundaries. To render this precise; the paper first describes cl_∞ - monoids; which are used to measure the degree of membership of points in sets. Then L- or fuzzy sets are defined; and suitable collections of these are called L-Topological spaces.

A number of examples and result for such space are given. Perhaps most interesting is a version of the Tychonoff theorem which given necessary and sufficient conditions on L for all collection with given cardinality of compact L- spaces to have compact product.

We know Tychonoff Theorem as Arbitrary products of compact spaces are compact. Introduction-There seems to be use in certain infinite valued logics [2] in constructive analysis [1]; and in mathematical psychology [6] for notions of proximity less restrictive than those found in ordinary topology

This paper develops some basic theory for spaces in which open sets are fuzzy. First we need sufficiently general sets of truth values with which to measure degree of membership. The main thing is no get enough algebraic structure.

1.1 Definition- A cl_∞ - monoid is a complete lattice L with an additional associative binary operation $*$ such that the lattice zero 0 is a zero for $*$. The lattice infinity 1 is a identity for $*$ and the complete distributive laws

$$a * \left[\bigvee_{i \in I} b_i \right] = \bigvee_{i \in I} (a * b_i)$$

And

$$\left[\bigvee_{i \in I} b_i \right] * a = \bigvee_{i \in I} (b_i * a)$$

Hold for all $a_i\ b_i \in L$; and all index sets I.

1.2 Definition- An Let A on a set X is a function $A{:}X \rightarrow L$. X is called the carrier of A : L is called the truth set of A; and for $x \in X$; A(x) is called the degree of membership of x in A.

1.3 Definition- An L-topological (or just L-) space is a pair <X; œ > such that X is a set ; œ $\subseteq L^X$ and

(1) ƹ ⊆ œ ⇒ ∨ƹ ∈ œ
(2) A: B ∈ œ ⇒ A ∗ B ∈ œ
(3) O_X ; 1_X ∈ œ .

1.4 Theorem of Fuzzy Tychonoff Theorem- For solving fuzzy Tychonoff theorem we firstly define the following lemma.

1.5 Lemma- For a:b elements of a cl_∞ - monoid L;

$$a * b \leq a \wedge b.$$

Proof- $a \leq 1$, so $1 * b = (a \vee 1) * b$
$= a * b \vee 1 * b$

That is $b = (a * b) \vee b$;
and therefore $a * b \leq b$.
similarly $a * b \leq a$.

1.6 Theorem- Every product of α -compact L-spaces is compact iff 1 is α-isolated in L."

Proof – with the help of theorem, if φ is a sub-base for <x: œ > and every cover of X by sets in φ has a finite sub-cover; then < X. œ > is compact.

Now we have to proof fuzzy-Tychonoff theorem :-

We have < X_i œ$_i$ > compact for $i \in I$; $|I| \leq \alpha$ and

< x: œ > = $\pi_{i \in I}$. <X_i: œ$_i$ >.

Where œ has the sub-base.

φ = {p^1_i (A_i)| A_i ∈ œ i; i ∈ I }

So by the above theorem it suffices to show that no Ӎ ⊆ φ with FUP (finite union property) is a cover.

Let Ӎ ⊆ φ with F U P be given: and for i∈I let Ӎ$_i$ = { A∈ œ$_i$ | P_i^{-1} (A) ∈ Ӎ }. Then each Ӎ$_i$ has FUP;

For if $\vee^{n}_{j=1}\ A_j = 1xi$;
Where $P_i^{-1}(A_j) \in Ŭ$
Then n
$V_{j=1}\ P^{-1}_i (A_j) = 1x$
Since $P_i^{-1}(1xi) = 1x$ and P^{-1}_i preserves
Therefore $Ŭ_i$ is a non-cover for each $i \in I$ and there exits $x_i \in X_i$ such that $(V Ŭ_i)(x_i) = a_i < 1$
Now let $x = < x_i > \in X$ and
$Ŭ_I^{'} = \{ P^{-1}_i (A) | A \in œ_i \} \cap Ŭ$
Then $Ŭ \subseteq \varphi$ implies $Ŭ = \cup_I Ŭ_I^{'}$ and
$(V Ŭ_i)(xi) = a_i$ implies
$(V Ŭ_i)(x) = a_i$
Since $(V Ŭ_I^{'})(x) = V \{ (P_i^{-1}(A))(x) | A \in œ_i$
and $P_i^{-1}(A) \in Ŭ \}$

$= V \{ A (P_i(x)) | P_i^{-1}(A) \in Ŭ$
and $A \in œ_i \} = (V Ŭ_i)^a (xi)$.
Therefore $(V Ŭ)(x) = \underset{i \in I}{V} ((V Ŭ^{'}_I)(x))$
$= \underset{i}{V}\ ai < 1$
since I is α- isolated in I; each
$a\,i < 1$ and $|I| \leq \alpha$.

1.7 Theorem- If 1 is not α –isolated in L; then there is a collection $<xi: œ_i>$ for $i \in I$ and $|I| = \alpha$ of compact L-spaces such that the product $<x:œ>$ is non-compact.

Proof- non α- isolation given I with $|I| = \alpha$ and $ai < 1$ for $i \in I$ such that $V_i\ a_i = 1$.

Let $X_i = \mathbb{N}$ for $i \in I$ and
Let $œ_i = \{ 0, (ai)_{[n]}. 1 \}^*$; where
As ; for $a \in L$ and $S \subseteq Xi$; denote the function equal to a on S and O or X_i-S; where $[n] = \{ 0,1 \ldots\ldots\ldots\ldots\ldots N-1\}$ and where $\varphi *$ indicates the L- topology generated by φ as a sub-basis.

Then $< \mathbb{N}, œ_i>$ is a compact. As every $A \in œ_i$ except 1 is contained in $(a_i)_N$; and $(a_i)_N < 1$; that
$Ŭ \subseteq œ_i$ is a cover iff $1 \in Ŭ$ therefore every cover has {1} as a sub-cover.
But $<x : œ> = \pi_{i \in I} <X_i : œ_i>$ is non compact. Let A_{in} denote the open L-set $P_i^{-1}((ai)[n]) \in œ$: for $i \in I$
at is given for $x \in X$ by the formula
$A_{in}(x) = a\,i$ if $x_i < n$;
and $A_{in}(x) = O$ otherwise

$\{ A_{in}\}\ i \in I; n \in \mathbb{N} \}$
is an open cover with no finite sub-cover.

First $V_n\ A_{in} = (a_i)_x$. since
$V_n(ai)_{[N]} = (ai)_{\mathbb{N}}$ in $<xi:œ_i;>$
and P_i^{-1} preserves suprema so that
$V_{in}A_{in} = Vi\ (V_n\ A_{in}) = Vi\ (a_i)_x = 1$
Second no finite sub-collection
$\{A_{in} | < i;n> \in J \}$; $|J| < \omega$ can
Cover; since $V<i,n> \in J\ A_{in}(x) = O$
For any x with
$X_i \geq V \{n| <i; n> \in J \}$ For any i
and infinitely many such x always exist.

1.8 Initial and Final Topologies- In this paper we introduce the notations of initial and final topologies. For this firstly we defined initial and final topologies then their theorems. We show that from a categorical point of view they are the right concept of generalize the topological ones.

We show that with our notion of fuzzy compactness the Tychonoff product theorem is safeguarded. We also show that this is not the case for weak fuzzy compactness.

1.9 Initial Fuzzy Topologies- If we consider a set E; a fuzzy topological space (f:y) and a function $f : E \rightarrow F$ then we can define
$f^{-1}(Y) = \{ f^{-1}(ג) : ג \in Y \}$
It is easily seen that $f^{-1}(Y)$ is the smallest fuzzy topology making f fuzzy continuous.
More generally: consider a family of fuzzy topological space
$(F_j : Y_j)\ j \in J$ and for each $j \in J$ a function
$f_j : E \rightarrow f_j$
then it is easily seen that the union

$$\bigcup_{j \in J} f^{-1}j(Y_j)$$

is a sub-base for a fuzzy topology on E making every f_i fuzzy continues . we denote it by

$$f^{-1}_j (Y_j) \Big\langle \bigcup_{j \in J} f^{-1}_j\ (Y_j) \Big\rangle_{j \in J} \text{ or sup}$$

1.10 Definition- The initial fuzzy topology on E for the family of fuzzy topological space (Fj: Yj) $j \in J$ and the family of functions

$$f_j : E \rightarrow (F_j : y_j)$$

is the smallest fuzzy topology on E making each function f_j fuzzy continuous.

Within the notations of the definition let (G:δ) be a fuzzy topological space and let f be a function from G to E then f will be fuzzy continuous iff all the compositions $f_j o f$ are fuzzy continues. Moreover the initial fuzzy topology on E is the finest one possessing this property; it is equally clear that the transitivity property for initial topologies holds for initial fuzzy topologies.

1.11 Definition- If (E_j, δ_j) $j \in J$ is a family of fuzzy topological space; then the fuzzy product topology on $\pi_{j\in J} E_j$ is defined as the initial fuzzy topology on $\pi_{j\in J} E_j$ for the family of spaces $(E_j. \delta_j)$ $j \in J$ and function fi: $\pi_{j\in J} E_j \to$ Ei

Where$\forall$ i $\in$ J

f i = P_{ri},

the protection on the i^{th} coordinate.

1.12 Definition- if $(E_j, \delta_j)_{j\in J}$ is a family of fuzzy topological space then the fuzzy product topology, on $\pi_{j\in J} E_j$ is defined as the initial fuzzy topology on $\pi_{j\in J} E_j$ for the family of spaces $(E_j,\delta_j)_{j\in J}$ and functions fi: $\pi_{j\in J}$ $\pi_{j\in J} E_j \to$ Ei where $\forall$ i $\in$J fi = p_{ri} the projection on i^{th} coordinate we denote this fuzzy topology $\pi_{j\in J} \delta_j$.

1.13 Theorems of Initial Topologies- The Theorems related to initial fuzzy topologies is as follows-

1.14 Lemma- if f: E → F, δ then

$$i\,(f^{-1}(\delta)) = f^{-1}(i(\delta)) = i\,(f^{-1}(\delta)).$$

Proof - f:E; f^{-1} (i (δ)) → F.

i (δ) is continuous: thus f: E; W (f^{-1}(i(δ)))→ F;

δ is fuzzy continuous.

This implies W (f^{-1}(i(δ))) ⊃ f^{-1} (δ) ⊃ f^{-1} (δ) ⇒ f^{-1} (i(δ)) ⊃ i (f^{-1} (δ))) ⊃ i (f^{-1} (δ)).

Further: f: E ; f^{-1} (δ) → F

δ is fuzzy continuous thus

f E : i (f^{-1}(δ)) → F.

i (δ) is continuous and this implies

i (f^{-1}(δ)) ⊃ f^{-1} (i(δ)).

1.15 Theorem- If E is a set $(F_j: T_j)_{j\in J}$ is a family of topological space and

$f_j : E \to E_j$ is a family of functions then we have

$$\sup_{j\in J} f^{-1}_j (w(T_j)) = w (\sup_{j\in J} f_j^{-1}(T_j))$$

Proof- we can prove this theorem By the following lemma:-

Lemma 1:-if f: E → F : δ then

$$i\,(f^{-1}(\delta)) = f^{-1}(i(\delta)) = i\,(f^{-1}(\delta)).$$

Lemma 2:-if f: E → F: W (T) then

$$W\,(f^{-1}(T)) = f^{-1}(W(T)).$$

Consider now a set E; a family of fuzzy topological spaces $(F_j: \delta_j)_{j\in J}$ and a family of functions $f_j : E \to F_j$

Then on the one hand: we can consider the initial fuzzy topology as defined and take the associated topology

i.e. $i\,(\sup_{j\in J})\, f^{-1} (\delta_j))$

on the other hand we can consider the family of topological spaces $(F_j;\ i\ (\delta_j))_{j\in J}$ and take the initial topology i.e.

$$\sup_{j \in J} f^{-1}_j (i(\delta_j)).$$

Showing that these two topologies are equal is trivial.

1.16 Theorem- if E is a set $(F_j. \delta_j)_{j\in J}$ is a family of fuzzy topological spaces: and

$f_j : E \to F_j$ is a family of function: then we have

$$i\,(\sup_{j\in J} f^{-1}_j(\delta_j)) = \sup_{j\in J} f_j^{-1}(i(\delta_j)).$$

Proof- we can prove this theorem by the help of lemma 1

if f: E → F: δ then

$i\,(f^{-1}(\delta)) = f^{-1}(i(\delta)) = i\,(f^{-1}(\delta))$.

It follows that $\forall_{j\in J}$

$f_j^{-1}(i(\delta_j)) = i\, f^{-i}_j(\delta_j)$.

and the result then follows analogously by the lemma 2

Lemma 3:- if E is a set and $(T_j)_{j\in J}$ is a family of topologies on E then

$$\sup_{j\in J} \omega(T_j) = \omega(\sup_{j\in J} T_j).$$

1.17 Final Fuzzy Topologies- Consider a set F a fuzzy topological space (E:δ): and a function

f: E → F

Then we can define

$f(\delta) = <\{\vee : f^{-1}(\vee) \in \delta$

$\}>$

more generally consider a family of fuzzy topological space $(E_j: \delta_j)_{j\in J}$ and for each $j\in J$ a function

$f_j : E_j \to F$.

It is easily seen that the intersection

$$\bigcap_{j\in J} f_j(\delta_j)$$

is the finest fuzzy topology on F making all the functions f_j fuzzy continuous.

1.18 Definition- the final fuzzy topology on F: for the family of fuzzy topological spaces $(E_j:\delta_j)_{j\in J}$ and the family of functions

$f_j: (E_j:\delta_j) \to F$

is the finest fuzzy topology on F making each functions f_j: fuzzy continuous.

And $\forall_j \in J$:

Let $\delta_j = \omega(T_j)$

Then on the other hand we have

$$\bigcap_{j\in J} f_j(\omega(T_j)).$$

and on the other hand we have

$$W\,(\bigcap_{j\in J} f_j\,(T_j))$$

again we can show the two fuzzy topological to be the same.

1.19 Lemma 1- if f:E δ → F

Then we have $i\,(f(\delta)) = f(i(\delta)) \supset i\,(f(\delta))$.

Proof- to show the inclusions

$i\,(f(\delta)) \supset f\,(i(\delta)) \supset i\,(f(\delta))$

we proceed as we proceeded in lemma:-

if f: E → F: δ then

$i\;(f^{-1}(\delta)) = f^{-1}\;(i(\delta)) = i\,(f^{-1}(\delta))$.

The remaining inclusion is shown using the latter one upon δ.

i.e $i\;(f(\delta)) \subset f(i(\delta)) = f(i o\omega\,(i(\delta))) = f\,(i(\delta))$.

1.20 Lemma 2- if f:E: ω (T) → F then we have

$f\,(\omega(T)) = \omega\,(f\,(T))$.

Proof- from lemma as proved above. at follows with

δ = ω (T) then

$\overline{f\,(\omega\,(T))} = W\,(f\,(T))$

However: f (ω (T) is topologically generated indeed it is the finest fuzzy topology making

f: E : ω (T) → F:

f (W(T)) fuzzy continuous

But since ω (T) is topologically generated

f: E: ω (T) → F:

f (ω(T)) still is fuzzy continuous and thus

f (ω(T) = f (w(T)).

1.21 Conclusion- In this section we discussed about the fuzzy Tychonoff theorems and initial and final fuzzy topology fuzzy Tychonoff theorems is describes as " arbitrary products of compact spaces are compact". We show that with our notion of fuzzy compactness the Tychonoff product theorem is safeguarded.

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