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Existence Of Solutions For Nonlinear Difference Equations With Nonlocal Conditions

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Abstract

Existence of mild solutions of initial value problem for nonlinear difference equation with nonlocal conditions is obtained. The proof relay on the fixed point theorem for compact mapping.

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1. Introduction:

Due to wide application in many fields such as science, economics, neural network, ecology, the theory of nonlinear difference equations has been widely studied since 1970; see, for example, [1, 3, 15, 16]. At the same time, boundary value problems and initial value problems of difference equations have received much attentions from many authors; see [2,4,7,9,10,13,19].

In this paper, we consider an initial value problem for a nonlinear difference equation with nonlocal initial conditions. More precisely we consider the IVP

$$\Delta y(t) = A(t, y(t))y(t) + f(t, y(t)) \qquad (1.1)$$

$$y(0) + g(y) = y_0 \qquad (1.2)$$

where $t \in J = \{0, 1, 2, \ldots, N\}$, $A, f : J \times E \to E$ are two continuous functions, $g : C(J, E) \to E$, $y_0 \in E$ and E is a real Banach space with the norm $\|\,.\,\|$. The existence of solutions for IVP (1.1)-(1.2) is obtained by using the classical fixed point theorem for compact map due to Smart [18].

Such problems with classical initial conditions have been studied by Anichini [5], Anichini and Conti [6], Conti [11], Conti and Iannacci [12], Kartsatos [14], Mario and Pietramala [17].

The nonlocal conditions, which are generalization of the classical initial conditions was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski [8].

2. Preliminary Notes and Hypothesis

Let E be a real Banach space and $J = \{0, 1, 2, \ldots, N\}$, $C(J, E)$ denote the Banach space of functions $y : J \to E$ equipped with the norm

$$\|y\|_\infty = sup\{\|y(t)\| : t \in J\}$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E with norm

$$\| T \|_{B(E)} = sup\{\| T(y) \| : \| y \| = 1\}.$$

Let us list the following hypothesis:

(H_1) $A : J \times E \to B(E)$, $(t, v) : \to A(t, v)$ is a continuous function with respect to v such that for $r > 0$ there exists $r_1 > 0$ such that

$$\|v\| \le r \to \|A(t, v)\|_{B(E)} \le r_1$$

for all $t \in J$ and $v \in E$.

(H_2) $f : J \times E \to E$, $(t; v) : \to f(t; v)$ is a continuous function with respect to v.

(H_3) A function $g : C(J, E) \times E$ is continuous and there exists a constant $L > 0$ such that $\| g(y) \| \le L$ for each $y \in E$.

(H_4) There exists constant K such that $\| f(t, u) \| \le K$ for all $t \in J$, $u \in E$.

For each $u \in C(J, E)$, define a function $U_u : J \times J \to B(E)$ such that

$$U_u(t, s) = I + \sum_{w=s}^{t-1} A_u(w) U_u(w, s) \qquad (2.1)$$

where I stands for an identity operator on E and $A_u(t) = A(t, u(t))$.

From (2.1), one has

$$U_u(t, s) = I,\ s \le t, u \in E \qquad (2.2)$$

and

$$\Delta_t U_u(t,s) = A(t,u(t))U_u(t,s) \qquad (2.3)$$

In the list of hypothesis, assume

(H_5) $M = sup\{\|U_y(t, s)\|_{B(E)} : (t, s) \in J \times J, y \in B(E)\}$.

Remark 2.1 From (H_1), it follows that for $u \in C(J, E)$, $A_u \in C(J, B(E))$.

Remark 2.2 Suppose $\{u_n\}$ is a sequence in $C(J, E)$ converging to $u_* \in C(J, E)$. Then (H_1) implies $Au_n \to Au_*$ i.e. $A(t, u_n(t)) \to A(t, u_*(t))$ for all $t \in J$.

Now we prove the following lemma.

Lemma 2.1 If (H_1) holds then for each $u \in C(J, E)$; $U_u : J \times J \to B(E)$ defined by (2.1) is continuous with respect to u.

Proof: From (2.1) by putting $t = s+1$ and using $U_u(s, s) = I$, we get

$$U_u(s+1, s) = I + A(s, u(s))U_u(s, s) = I + A(s, u(s)).$$

For $t = s+2$,

$$U_u(s+2, s) = I + A(s, u(s))U_u(s, s) + A(s+1, u(s+1))U_u(s+1, s) = (I + A(s, u(s)))(I + A(s+1, u(s+1))).$$

Continuing the process, we obtain

$$U_u(t,s) = \prod_{w=s}^{t-1}(I + A(w,u(w))) \qquad (2.4)$$

Now, suppose $\{u_n\}$ is a sequence in $C(J, E)$ converging to $u_* \in C(J, E)$. From (2.4), one has

$$U_{u_n}(t,s) = \prod_{w=s}^{t-1}(I + A(w,u_n(w)))$$

and

$$U_{u_*}(t,s) = \prod_{w=s}^{t-1}(I + A(w,u_*(w)))$$

The conclusion follows from (H_1) and Remark 2.2.

Theorem 2.1: A function $y \in C(J, E)$ given by

$$y(t) = U_y(t,0)y_0 - U_y(t,0)g(y) + \sum_{s=1}^{t} U_y(t,s) f(s-1, y(s-1)) \qquad (2.5)$$

is a solution of IVP (1.1)-(1.2).

Proof: It follows from (2.2) and (2.5) that

$$y(0) = U_y(0, 0)y_0 - U_y(0, 0)g(y)$$

i.e. $y(0) + g(y) = y_0$.

From (2.3), we have

$\Delta\, U_y(t, 0) = A(t, y(t))Uy(t, 0)$

and

$\Delta\, U_y(t, s) = A(t, y(t))U_y(t, s)$.

Put

$$z(t) = \sum_{s=1}^{t} U_y(t,s) f(s-1, y(s-1)).$$

Therefore,

$$\Delta z(t) = \sum_{s=1}^{t}\{\Delta_t U_y(t,s) f(s-1, y(s-1))\} + U_y(t+1,t+1) f(t, y(t))$$

$$= A(t, y(t)) \sum_{s=1}^{t} U_y(t,s) f(s-1, y(s-1)) + f(t, y(t)).$$

Thus equation (2.5) reduces to

$$y(t) = U_y(t, 0)y_0 - U_y(t, 0)g(y) + z(t).$$

Therefore,

$$\Delta\, y(t) = \Delta\, U_y(t, 0)y_0 - \Delta\, U_y(t, 0)g(y) + \Delta\, z(t) = A(t, y(t))y(t) + f(t, y(t)).$$

This completes the proof.

A function $y(t)$ given by (2.5) is called mild solution of IVP (1.1)-(1.2).

3. Existence Theorems:

In this section we establish the existence of solution of IVP (1.1)-(1.2).

The following Lemma is crucial in the proof of main theorem.

Lemma 3.1: (18) Let X be a Banach space and let $T : X \to X$ be a continuous compact map. If the set $\Omega = \{y \in X : \lambda y = T(y), \text{ for some } \lambda > 1\}$ is bounded, then T has a fixed point.

Theorem 3.1 : Assume that hypothesises (H_1)-(H_5) are satisfied. Then the problem (1.1)-(1.2) has at least one mild solution on J.

Proof: We transform the problem (1.1)-(1.2) into a fixed point problem. Consider the mapping $T : C(J, E) \to C(J, E)$ defined by

$$(Ty)(t) = U_y(t,0)y_0 - U_y(t,0)g(y) + \sum_{s=1}^{t} U_y(t,s)\, f(s-1, y(s-1)). \qquad (3.1)$$

It is clear that the fixed points of T are mild solutions of (1.1)-(1.2).

We shall show that T is a continuous compact mapping. The continuity of T follows from Lemma 2.1 and hypothesises (H_2), (H_3). Now we prove that T maps bounded sets into relatively compact sets. i.e. T is a compact mapping. Let $B_r = \{y \in C(J, E) : \|y\| \le r\}$.

Then for each $t \in J$ and $y \in B_r$, we have

$$(Ty)(t) = U_y(t,0)y_0 - U_y(t,0)g(y) + \sum_{s=1}^{t} U_y(t,s)f(s-1, y(s-1)).$$

By (H_3)-(H_5) we have

$$\|Ty\| \le \|U_y(t,0)\|\|y_0\| + \|U_y(t,0)\|\|g(y)\| + \sum_{s=1}^{t}\|U_y(t,s)\|\|f(s-1, y(s-1))\|$$

$$\le M(\|y_0\| + L + KN).$$

Therefore T is bounded on B_r. Now for $t_1, t_2 \in J$ and $t_1 < t_2$, using (2.3) we obtain

$$U_y(t_2, s) - U_y(t_1, s) = U_y(t_2, s) - U_y(t_2-1, s) + U_y(t_2-1, s) - \ldots\ldots + Uy(t_1+1, s) - U_y(t_1, s)$$

$$= \sum_{w=t_1}^{t_2-1} A(w, y(w))U_y(w,s) \quad (3.2)$$

By (H_1), (H_3) - (H_5) and (3.2) we have

$$\|Ty(t_2) - Ty(t_1)\| \le \|U_y(t_2, 0) - U_y(t_1, 0)\|(\|y_0\|+L) + \sum_{s=1}^{t_1}\|U_y(t_2,s) - U_y(t_1,s)\|\|f(s-1, y(s-1))\| + \sum_{s=t_1+1}^{t_2}\|U_y(t_2,s)\|\|f(s-1, y(s-1))\|$$

$$\le r_1M(\|y_0\|+L)(t_2-t_1) + r_1MKN(t_2-t_1) + MK(t_2-t_1)$$

$$= M[r_1(\|y_0\|+L) + K(r_1N+1)](t_2-t_1)$$

$$\le K_1(t_2-t_1),$$

for some K_1. Hence $T(B_r)$ is an equicontinuous family of functions. Therefore by the Ascoli-Arzela theorem, $T(B_r)$ is relatively compact.

Now we prove that the set

$\Omega = \{y \in C(J, E) : \lambda y = T(y)$, for some $\lambda > 1\}$

is bounded. Let $y \in \Omega$, then $\lambda y = T(y)$ for some $\lambda > 1$. Therefore,

$$y(t) = \lambda^{-1}U_y(t,0)y_0 - \lambda^{-1}U_y(t,0)g(y) + \lambda^{-1}\sum_{s=1}^{t} U_y(t,s)f(s-1, y(s-1))$$

By (H_3)-(H_5), for each $t \in J$, we have

$$\|y(t)\| \le \|U_y(t,0)\|(\|y_0\|+L) + \sum_{s=1}^{t}\|U_y(t,s)\|\|f(s-1, y(s-1))\|$$

$$\le M(\|y_0\| + L + KN).$$

Therefore, $\|y\| = \sup\{\|y(t)\| : t \in J\} \le K_2$ for some $K_2 > 0$. This shows that Ω is bounded. Set $X = C(J, E)$. As a consequence of Lemma 3.1, we deduce that T has a fixed point which is a mild solution of (1.1)-(1.2).

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