# Some Results Concerning Fixed Point In Hilbert Spaces

**Shailesh T. Patel, Sunil Garg, Dr.Ramakant Bhardwaj**
The student of Singhania University
Sinior Scientist,MPCST Bhopal
Truba Institute of Engineering & Information Technology,Bhopal(M.P.)

## ABSTRACT

**In the present paper we will find some results concerning fixed point and common fixed point theorems in Hilbert spaces for rational expression, which will satisfy the well known results.**

**Key words:** Common Fixed Point, Hilbert spaces

## INTRODUCTION

PaSndhare and Waghmode [3] proved the following result in Hilbert space The study of properties and application of fixed point of various type of contractive mapping in Hilbert spaces were obtained among others by Browder, F.E.,and Petryshyn [1], Hicks,T.L. and Huffman,Ed.W'[2], Huffman[3],Koparde and Waghmode[4], Smita Nair and Shalu Shrivastava[5,6].In this paper we present a common fixed point theorem involingself mapping.

The study of properties and applications of fixed point of various type of contractive mapping in also Banach spaces were obtained among others by Browder, F.E.[7], Browder, F.E.,and Petryshyn, W.V.[8].

Theorem A Let C be a close subset of Hilbert spaceX and let T be a self mapping on C satisfying.

$$\|Tx-Ty\|^2 \le a[\|x-Tx\|^2+\|y-Ty\|^2]+b\|x-y\|^2$$

For all x,y$\epsilon$C, $x\neq y$,0 < b, 0 <1 and 2a+b<1.

Then T has a unique fixed point.

In the present paper , we first extend Theorem A as follows:

Theorem**3.**1Let C be a closed subset of Hilbert space X and T be a mapping on X into it self satisfying

$$\|Tx-Ty\|^2 \le a\|x-Tx\|^2 + \beta\|y-Ty\|^2 + \gamma\|x-y\|^2$$

$$+\delta\min\{\|x-Ty\|^2,\|y-Tx\|^2\}+\eta\frac{\|y-Tx\|^2}{1+\|x-Tx\|\|x-Ty\|} \quad .........3.1.1$$

For all x,y in X where α,β,γ and δ are non-negative real with α+β+γ+4δ<1.Then T has a unique fixed point in X.

Proof : Let $x_0 \epsilon C$ and arbitrary we define a sequence $\{x_n\}$ as follows:

$x_1=Tx_0$, $x_2=Tx_1=T^2x_0 \ldots x_n=Tx_{n-1}=T^n x_0$, $x_{n+1}=Tx_n=T^{n+1}x_0$

If for some n,$x_{n+1}=x_n$,then it immediately follows $x_n$ is a fixed point of T, i.e.$Tx_n=x_n$

We suppose that $x_{n+1}\neq x_n$ for every n=0,1,2…..,Then appling (3.1.1),

We have for all n≥1

$$\|x_{n+1}-x_n\|^2 = \|Tx_n - Tx_{n-1}\|^2$$

$$\le a\|x_n - Tx_n\|^2 + \beta\|x_{n-1}-Tx_{n-1}\|^2 + \gamma\|x_n - x_{n-1}\|^2 +$$

$$\delta\min\{\|x_n-Tx_{n-1}\|^2,\|x_{n-1}-Tx_n\|^2\}+\eta\frac{\|x_{n-1}-Tx_n\|^2}{1+\|x_n-Tx_n\|\|x_n-Tx_{n-1}\|} \quad .........3.1.1$$

$$\le a\|x_n - x_{n+1}\|^2 + \beta\|x_{n-1}-x_n\|^2 + \gamma\|x_n - x_{n-1}\|^2 +$$

$$\delta \min\{\|x_n - x_n\|^2, \|x_{n-1} - x_{n+1}\|^2\} + \eta \frac{\|x_{n-1} - x_{n+1}\|^2}{1 + \|x_n - x_{n+1}\|\|x_n - x_n\|}$$

$$\leq a\|x_n - x_{n+1}\|^2 + \beta\|x_{n-1} - x_n\|^2 + \gamma\|x_n - x_{n-1}\|^2 + \eta\left[\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2\right]$$

$$\leq (\beta + \gamma + \eta)\|x_{n-1} - x_n\|^2 + (\alpha + \eta)\|x_n - x_{n+1}\|^2$$

$$(1 - \alpha - \eta)\|x_{n+1} - x_n\|^2 \leq (\beta + \gamma + \eta)\|x_{n-1} - x_n\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq \frac{(\beta + \gamma + \eta)}{(1 - \alpha - \eta)}\|x_{n-1} - x_n\|^2$$

$$\|x_{n+1} - x_n\|^2 \leq k\|x_{n-1} - x_n\|^2$$

where k= $\frac{(\beta + \gamma + \eta)}{(1 - \alpha - \eta)}$

→0 as n→∞

$$\|x_{n+2} - x_{n+1}\|^2 = \|Tx_{n+1} - Tx_n\|^2$$

$$\leq a\|x_{n+1} - Tx_{n+1}\|^2 + \beta\|x_n - Tx_n\|^2 + \gamma\|x_{n+1} - x_n\|^2$$

$$+\delta \min\{\|x_{n+1} - Tx_n\|^2, \|x_n - Tx_{n+1}\|^2\} + \eta \frac{\|x_n - Tx_{n+1}\|^2}{1 + \|x_{n+1} - Tx_{n+1}\|\|x_{n+1} - Tx_n\|}$$

$$\leq a\|x_{n+1} - x_{n+2}\|^2 + \beta\|x_n - x_{n+1}\|^2 + \gamma\|x_{n+1} - x_n\|^2$$

$$+\delta \min\{\|x_{n+1} - x_{n+1}\|^2, \|x_n - x_{n+2}\|^2\} + \eta \frac{\|x_n - x_{n+2}\|^2}{1 + \|x_{n+1} - x_{n+2}\|\|x_{n+1} - x_{n+1}\|}$$

$$\leq a\|x_{n+1} - x_{n+2}\|^2 + \beta\|x_n - x_{n+1}\|^2 + \gamma\|x_{n+1} - x_n\|^2$$

$$+\eta\{\|x_n - x_{n+1}\|^2 + \|x_{n+1} - x_{n+2}\|^2\}$$

$$\leq (\beta + \gamma + \eta)\|x_n - x_{n+1}\|^2 + (\alpha + \eta)\|x_{n+1} - x_{n+2}\|^2$$

$$\|x_{n+2} - x_{n+1}\|^2 \leq \frac{(\beta + \gamma + \eta)}{(1 - \alpha - \eta)}\|x_n - x_{n+1}\|^2$$

$$\|x_{n+2} - x_{n+1}\|^2 \leq k^2\|x_{n-1} - x_n\|^2$$

where k= $\frac{(\beta + \gamma + \eta)}{(1 - \alpha - \eta)}$

Proceeding in this way, we obtain

$$= k^n\|x_0 - x_1\|$$

n=1,2,……..

Where α+β+γ+2η<1, because k < 1

$$\|x_n - x_{n+1}\|^2 \leq k^n\|x_0 - x_1\|^2$$

Hence for any positive integer p,

$$\|x_n - x_{n-1}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \ldots\ldots + \|x_{n+p-1} - x_{n+p}\|$$

$$\leq (k_n + k_{n+1} + \ldots\ldots\ldots + k_{n+p-1})\|x_{n-1} - x_n\|$$

$$= \frac{k^n}{1-k}\|x_{n-1} - x_n\|$$

in 0<k<1

Thus $\|x_n - x_{n+p}\| \to 0$ as $n \to \infty$.

Therefore $\{x_n\}$ is a **Cauchy sequence** in C. Since C is a closed subset of a Hilbert space X ,there exists an element u Є C which is the limit of $\{x_n\}$.

i.e. u = $\lim_{n\to\infty} x^n$ Now, further we have,

$$\|u-Tu\|^2 = \|(u-x_n)+(x_n-T_u)\|^2$$

$$\le \|u-x_n\|^2 + \|x_n - T_u\|^2 + 2\operatorname{Re}\langle u-x_{n,}x_n - Tu\rangle$$

$$\le \|u-x_n\|^2 + \alpha\|x_{n-1} - Tx_{n-1}\|^2 + \beta\|u-Tu\|^2 + \gamma\|x_{n-1}-u\|^2$$

$$+\delta\min\{\|x_{n-1}-Tu\|^2, \|u-Tx_{n-1}\|^2\} + \eta\frac{\|Tx_{n-1}-u\|^2}{1+\|Tx_{n-1}-x_{n-1}\|\|x_{n-1}-Tu\|}$$

$$+2\operatorname{Re}\langle u-x_{n,}x_n - Tu\rangle$$

Letting $n\to\infty$, so that $x_{n-1}, x_n \to u$, we get

$$\|u-Tu\|^2 \le \beta\|u-Tu\|^2$$

This implies u= Tu, since $\beta<1$

If follows that **u is a fixed point** of T. **We now show that u is unique** ,

For that let v ϵ C another common fixed point T such that $v \ne u$ , then

$$\|v-u\|^2 = \|Tv-Tu\|^2$$

$$\le \alpha\|v-Tv\|^2 + \beta\|u-Tu\|^2 + \gamma\|v-u\|^2$$

$$+\delta\min\{\|v-Tu\|^2, \|u-Tv\|^2\} + \eta\frac{\|u-Tv\|^2}{1+\|v-Tv\|\|v-Tu\|}$$

$$\le (\gamma+\delta+\eta)\|v-u\|^2$$

This implies $\|v-u\|^2 \le (\gamma+\delta+\eta)\|v-u\|^2$

Since $\gamma+\delta+\eta < 1$

Remark: On taking α=β=a, γ=b, and δ=0, η=0 in theorem 1, we get theorem A.

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