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Anti Q-Fuzzy KU - Ideals In KU- Algebras And Its Lower Level Cuts

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Abstract

In this paper, we introduce the concept of Anti Q-fuzzy KU-ideals of KU-algebras, lower level cuts of a fuzzy set and prove some results . We show that a Q-fuzzy set of a KU-algebra is a KU-ideal if and only if the complement of this Q-fuzzy set is an anti Q-fuzzy KU-ideal. Also we discussed few results of Anti Q-fuzzy subalgebras of KU-algebra.

Keywords:- KU-algebra,fuzzy KU- ideal,Anti Q-fuzzy KU-ideal, Q-fuzzy subalgebra, Anti Q- fuzzy subalgebra, lower level cuts.

AMS Subject Classification (2000): 20N25, 03E72, 03F055 , 06F35, 03G25.

1.Introduction

Y.Imai and K.Iseki introduced two classes of abstract algebras : BCK-algebras and BCI – algebras**[6,7].** It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Q.P.Hu and X .Li introduced a wide class of abstract BCH-algebras**[4,5]**. They have shown that it is the class of BCI-algebras. J.Neggers, S.S.Ahn and H.S.Kim introduced Q-algebras which is generalization of BCK / BCI algebras and obtained several results**[10]**.W.A.Dudek and Y.B.Jun studied fuzzy ideals and several fuzzy structures in BCC-algebras are considered**[2,3]**. C.Prabpayak and U.Leerawat introduced a new algebraic structure which is called KU-algebras and investigated some properties**[11]**.Samy M.Mostafa and Mokthar A. Abdel Naby introduced fuzzy KU-ideals in KU-algebras**[14]**.R.Biswas introduced the concept of Anti fuzzy subgroups of groups**[1]**. Modifying his idea, in this paper we apply the idea of KU-algebras . We introduce the notion of Anti Q-fuzzy KU-ideals of KU-algebras and Anti Q-fuzzy subalgebras of KU-algebras.

2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1[11]

A non empty set X with a constant 0 and a binary operation $*$ is called a KU-algebra if it satisfies the following axioms.

1. $(x * y) *[(y * z) * (x * z)] = 0$
2. $0 * x = x$
3. $x * 0 = 0$
4. $x * y = 0 = y * x$ implies $x = y$, for all $x, y , z \in X$.

In X we can define a binary operation $\le$ by $x \le y$ if and only if $y * x = 0$. Then $(X,*,0)$ is a KU-algebra if and only if it satisfies that

i. $(y * z) * (x * z) \le (x * y)$
ii. $0 \le x$
iii. $x \le y$, $y \le x$ implies $x = y$.
iv. $x \le y$ if and only if $y * x = 0$, for all $x, y , z \in X$.

In a KU-algebra,the following identities are true**[14]**:

1. $z * z = 0$.
2. $z * (x * z) = 0$
3. $x \le y \Rightarrow y * z \le x * z$
4. $z * (y * x) = y * (z * x)$
5. $y * [(y * x) * x] = 0$, for all $x, y , z \in X$.

Example 2.1

Let $X = \{ 0 , a , b, c, d \}$ be a set with a binary operation $*$ defined by the following table

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	c
c	0	0	b	0	b
d	0	0	0	0	0

Then Clearly $(X , * , 0)$ is a KU-algebra.

Definition 2.2 [11]

Let $(X , * , 0)$ be a KU-algebra. A non empty subset I of X is called KU- ideal of X if it satisfies the following conditions

(1) $0 \in I$
(2) $x * (y * z) \in I$ and $y \in I \Rightarrow x * z \in I$
for all $x , y, z \in X$.

Remark: From Example 2.1, It is clear that $I_1 = \{0,a\}$ and $I_2 = \{0,a,b\}$ are KU-ideals of X.

Definition 2.3 [16]

Let X be a non-empty set**.** A fuzzy subset μ of the set X is a mapping $\mu : X \to [0, 1]$.

Definition 2.4 [8,9]

Let Q and G be any two sets.A mapping β: G x Q $\to [0, 1]$ is called a Q –fuzzy set in G.

Definition 2.5

A Q- fuzzy set μ in X is called a Q-fuzzy KU- ideal of X if

(i) $\mu(0,q) \geq \mu(x,q)$

(ii) $\mu(x * z, q) \geq \min\{\mu(x *(y * z),q), \mu(y,q)\}$,
for all x,y, z∈ X and q ∈ Q.

Definition 2.6 [9]

If μ be a Q-fuzzy set in set X then the complement denoted by μ^c is the Q-fuzzy subset of X given by $\mu^c(x,q) = 1-\mu(x,q)$, for all x,y∈ X and q ∈ Q.

3.ANTI Q-FUZZY KU-IDEALS

Definition 3.1

A Q-fuzzy set μ of a KU-algebra X is called an anti Q-fuzzy KU-ideal of X, if

(i) $\mu(0,q) \leq \mu(x,q)$

(ii) $\mu(x *z,q) \leq \max \{ \mu ((x * (y * z)) , q), \mu(y , q)\}$,
for all x,y,z∈ X and q ∈ Q.

Example 3.1

Let X = { 0 , 1 , 2, 3, 4 } be a set with a binary operation * defined by the following table

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	0	0	1	0

Then Clearly (X , * , 0) is a KU-algebra.
Let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 < t_1 < t_2$ .Define a Q-fuzzy set $\mu : X \times Q \to [0, 1]$ by $\mu(0,q) = t_0$, $\mu(1,q) = t_1 = \mu(2,q)$, $\mu(3,q) = t_2 = \mu(4,q)$,by routine calculations μ is an anti Q-fuzzy KU- ideal of X where q∈Q.

Theorem 3.1

Every Anti Q-fuzzy KU- ideal μ of a KU-algebra X is order preserving.

Proof

Let μ be an anti Q-fuzzy KU- ideal of a KU-algebra X and let x , y ∈ X and q ∈ Q.

Then $\mu(x,q) = \mu (0 * x , q)$
$\leq \max \{\mu (0 * (y * x) , q) , \mu (y,q)\}$
$= \max \{ \mu (0 * 0 , q) , \mu (y,q) \}$
$= \mu (y,q)$

$\mu(x,q) \leq \mu (y,q)$.

Theorem 3.2

μ is a Q-fuzzy KU-ideal of a KU-algebra X if and only if μ^c is an anti Q-fuzzy KU-ideal of X.

Proof :

Let μ be a Q-fuzzy KU- ideal of X and let x , y , z ∈ X and q ∈ Q.

(i) $\mu^c(0,q) = 1 - \mu(0,q)$
$\leq 1 - \mu (x,q)$
$= \mu^c (x,q)$

That is $\mu^c(0,q) \leq \mu^c (x,q)$.

(ii) $\mu^c (x * z, q) = 1 - \mu(x * z, q)$
$\leq 1 - \min \{ \mu (x * (y * z) , q) , \mu (y ,q) \}$
$= 1 - \min\{1- \mu^c (x * (y * z) , q) ,1 - \mu^c(y,q) \}$
$= \max \{ \mu^c (x * (y * z) , q) , \mu^c (y , q) \}$

That is,

$\mu^c (x * z , q) \leq \max \{ \mu^c (x* (y * z) , q) , \mu^c (y , q) \}$.

Thus μ^c is an anti Q-fuzzy KU-ideal of X. The converse also can be proved similarly.

Theorem 3.3

Let μ be an anti Q-fuzzy KU-ideal of KU – algebra X. If the inequality $x * y \leq z$, then $\mu(y,q) \leq \max\{\mu(x,q),\mu(y,q)\}$ for all x,y,z∈X and q∈Q.

Proof:

Assume that the inequality $x * y \leq z$ for all x,y,z ∈ X and q ∈ Q. Then $z * (x * y) = 0$.

Now, $\mu (y , q) = \mu (0 * y , q)$
$\leq \max \{ \mu (0 * (x * y) , q), \mu (x,q) \}$
$= \max \{ \mu (x * y,q), \mu (x,q)\}$
$\leq \max\{\max\{\mu (x * (z * y) , q), \mu(z , q)\} , \mu(x,q)\}$
$= \max\{\max\{\mu (z * (x * y) , q), \mu(z , q)\} , \mu(x,q)\}$
$= \max \{ \max \{ \mu (0, q), \mu(z , q)\} , \mu(x,q)\}$
$= \max \{ \mu(z , q)\} , \mu(x,q)\}$

$\therefore \mu (y,q) \leq \max \{\mu(x,q),\mu(y,q)\}$.

Theorem 3.4

If μ is an anti Q-fuzzy KU-ideal of KU–algebra X, then for all x, y ∈ X and q ∈ Q, $\mu (x * (x * y), q) \leq \mu (y , q)$

Proof:

Let x,y ∈ X and q ∈ Q.

$\mu (x*(x*y),q) \leq \max\{ \mu(x* (y*(x * y)), q), \mu(y, q) \}$
$= \max \{ \mu (x * (x * (y * y)), q), \mu(y, q) \}$
$= \max \{ \mu (x * (x * 0), q), \mu(y, q) \}$
$= \max \{ \mu (x * 0, q), \mu(y, q) \}$
$= \max \{ \mu (0, q), \mu(y, q) \}$
$= \mu (y,q)$

$\therefore \mu (x * (x * y), q) \leq \mu (y , q)$

4.LOWER LEVEL CUTS IN ANTI Q-FUZZY KU-IDEALS OF KU-ALGEBRA

Definition 4.1 [12]

Let μ be a Q-fuzzy set of X. For a fixed t ∈ [0, 1], the set $\mu^t =\{x \in X \mid \mu(x,q) \leq t$ for all $q \in Q\}$ is called the lower level subset of μ.

Clearly $\mu^t \cup \mu_t =X$ for t∈[0,1] if $t_1 < t_2$, then $\mu^{t1} \subseteq \mu^{t2}$.

Theorem 4.1

If μ is an anti Q-fuzzy KU-ideal of KU-algebra X,then μ^t is an KU-ideal of X for every t∈ [0,1] .

Proof

Let μ be an anti Q-fuzzy KU-ideal of KU-algebra X.

(i) Let $x \in \mu^t \Rightarrow \mu(x, q) \leq t$.
$\mu(0,q) = \mu (y * 0, q)$

$\leq$ max { μ (y * (x * 0),q) , μ(x,q) }
= max { μ (y * 0),q) , μ(x,q) }
= max { μ (0),q) , μ(x,q) }
= μ(x,q) $\leq$ t.

$\Rightarrow$ 0 ∈ μ^t .

(ii) Let x * (y * z) ∈ μ^t and y ∈ μ^t , for all x,y,z ∈ X and q ∈ Q.

$\Rightarrow$ μ (x * (y * z) ,q) $\leq$ t and μ (y,q) $\leq$ t.

μ ((x * z),q) $\leq$ max { μ ((x * (y * z)) , q), μ(y , q)}
$\leq$ max {t,t} = t.

$\Rightarrow$ x * z ∈ μ^t.

Hence μ^t is an KU- ideal of X for every t∈ [0,1].

Theorem 4.2

Let μ be a Q-fuzzy set of KU- algebra X.If for each t ∈ [0,1], the lower level cut μ^t is an KU-ideal of X, then μ is an anti Q- fuzzy KU-ideal of X.

Proof

Let μ^t be an KU-ideal of X.

If μ(0,q) > μ(x,q) for some x ∈ X and q ∈ Q.Then μ(0,q) > t_0 > μ(x,q) by taking $t_0 = \frac{1}{2}$ { μ(0,q) + μ(x,q) }.

Hence 0 ∉ μ^{t0} and x ∈ μ^{t0} , which is a contradiction.

Therefore, μ(0,q) ≤ μ(x,q).

Let x,y,z ∈ X and q ∈ Q be

μ((x * z) , q) > max{ μ (x * (y * z) , q), μ(y,q)}

Taking $t_1 = \frac{1}{2}$ { μ((x * z) , q) + max {μ (x * (y * z) , q), μ (y,q)}} and μ((x * z) , q) > t_1 > max{μ (x * (y * z) , q), μ(y,q)} .

It follows that (x * (y * z)),y ∈ μ^{t1} and x * z ∉ μ^{t1}.

This is a contradiction.

Hence μ((x * z), q) ≤ max{μ (x * (y * z) , q), μ(y,q)}

Therefore μ is an anti Q-fuzzy KU-ideal of X.

Definition 4.2 [13]

Let X be an KU- algebra and a,b ∈ X.We can define a set A(a,b) by A(a,b) ={ x ∈ X / a* (b* x) = 0 }.

It is easy to see that 0,a,b ∈ A(a,b) for all a,b ∈ X.

Theorem 4.3

Let μ be a Q-fuzzy set in KU-algebra X.Then μ is an anti Q- fuzzy KU- ideal of X iff μ satisfies the following condition.

$(\forall\, a,b \in X), (\forall\, t \in [0,1])$
$$(a,b) \in \mu^t \Rightarrow \mathrm{A(a,b)} \subseteq \mu^t$$

Proof:

Assume that μ is an anti Q-fuzzy KU- ideal of X.

Let a,b ∈ μ^t. Then μ (a,q) $\leq$ t and μ (b,q) $\leq$ t.

Let x ∈ A (a,b). Then a * (b * x) = 0.

Now,

μ (x,q) = μ (0 * x , q)
$\leq$ max { μ (0 * (b * x) , q) , μ (b,q)}
= max { μ ((b * x) , q) , μ (b,q)}
≤ max { max {μ (b * (a * x) , q) , μ (a,q)} , μ(b,q) }
= max { max {μ (a * (b * x) , q) , μ (a,q)} , μ(b,q) }
= max { max {μ (0,q) , μ (a,q)} , μ(b,q) }
= max { μ (a,q)} , μ(b,q) }≤ max { t , t} = t

$\Rightarrow$ μ (x,q) ≤ t .
$\Rightarrow$ x ∈ μ^t.

Therefore A(a,b) ⊆ μ^t .

Conversely suppose that A(a,b) ⊆ μ^t .

Obviously 0 ∈ A(a,b) ⊆ μ^t for all a,b ∈ X.

Let x,y,z ∈ X be such that x * (y * z) ∈ μ^t and y ∈ μ^t .

Since (x * (y * z)) * (y * (x * z)) = (x * (y * z)) * (x * (y * z)) = 0.

We have x * z ∈ A (x * (y * z) , y) ⊆ μ^t .

∴ μ^t is an KU- ideal of X.

Hence,by theorem 4.2, μ is an anti Q-fuzzy KU-ideal of X.

Theorem 4.4

Let μ be a Q-fuzzy set in KU-algebra X.If μ is an anti Q-fuzzy KU-ideal of X then
$$(\forall\, t \in [0,1])\ \mu^t \neq \emptyset \Rightarrow \mu^t = \bigcup_{a,b\, \in \mu^t} A(a,b).$$

Proof :

Let t ∈ [0,1] be such that $\mu^t \neq \emptyset$. Since 0 ∈ μ^t , we have $\mu^t \subseteq \bigcup_{a \in \mu^t} A(a,0) \subseteq \bigcup_{a,b \in \mu^t} A(a,b)$.

Now, let x ∈ $\bigcup_{a,b \in \mu^t} A(a,b)$.

Then there exists (u,v) ∈ μ^t such that x ∈ A (u,v) ⊆ μ^t by theorem 4.3. Thus $\bigcup_{a,b \in \mu^t} A(a,b) \subseteq \mu^t$.

∴ $\mu^t = \bigcup_{a,b \in \mu^t} A(a,b)$.

5.Anti Q- Fuzzy Sub algebra of KU algebra

Definition 5.1 [11]

A nonempty subset S of a KU-algebra X is said to be KU-subalgebra of X , if x , y ∈ S implies x * y ∈ S.

Definition 5.2

A Q-fuzzy set μ in a KU-algebra X is called a Q-fuzzy Subalgebra of X if μ(x * y , q) ≥ min{μ(x , q), μ(y , q)},for all x,y∈ X and q ∈ Q.

Definition 5.3 [15]

A Q-fuzzy set μ in a KU-algebra X is called an Anti Q-fuzzy sub algebra of X if μ(x * y , q) ≤ max{μ(x , q), μ(y , q)}, for all x,y∈ X and q ∈ Q.

Example 5.1

From example 3.1 , Let t_0, t_1, t_2 ∈ [0, 1] be such that $t_0 < t_1 < t_2$ .Define a Q-fuzzy set μ : X x Q →[0, 1] by μ(0,q) = t_0, μ(1,q) = t_1 = μ(2,q), μ(3,q) = t_2 = μ(4,q) ,By routine calculations μ is an anti Q-fuzzy subalgebra of X where q∈Q.

Theorem 5.1

If μ is an anti Q-fuzzy sub algebra of a KU-algebra X, then μ(0,q) ≤ μ (z,q) , for any z ∈ X and q ∈ Q.

Proof

Since z * z = 0 for any x ∈ X and q ∈ Q, then

μ(0, q) = μ (z * z, q)
≤ max {μ (z, q) , μ (z, q)} = μ (z,q)

∴ μ(0, q) ≤ μ (z, q).

Theorem 5.2

A Q-fuzzy set μ of a KU– algebra X is an anti Q-fuzzy subalgebra if and only if for every $t \in [0,1]$, μ^t is either empty or a KU-sub algebra of X.

Proof:

Assume that μ is an anti Q-fuzzy sub algebra of X and $\mu^t \neq \phi$. Then for any $x,y \in \mu^t$ and $q \in Q$, we have
$\mu(x * y,q) \leq \max\{\mu(x,q), \mu(y,q)\} \leq t$.
Therefore $x*y \in \mu^t$.Henceμ^t is a KU-sub algebra of X.

Now Assume that μ^t is a KU-sub algebra of X. Then for any $x,y \in \mu^t$ implies $x * y \in \mu^t$.
Take $t = \max\{\mu(x,q), \mu(y,q)\}$.
Therefore $\mu(x * y,q) \leq t = \max\{\mu(x,q), \mu(y,q)\}$.
Hence μ is an Anti Q-fuzzy sub algebra of X.

Theorem 5.3

Any KU-sub algebra of a KU– algebra X can be realized as a level KU-sub algebra of some Anti Q-fuzzy sub algebra of X.

Proof:

Let A be a KU-sub algebra of a given KU –algebra X and let μ be a Q-fuzzy set in X defined by

$$\mu(x,q) = \begin{cases} t, & if\ x \in A \\ 0, & if\ x \notin A \end{cases}$$

where $t \in [0,1]$ is fixed and $q \in Q$. It is clear that $\mu^t = A$.

Now we prove such defined μ is an anti Q-fuzzy sub algebra of X.
Let $x,y \in X$. If $x,y \in A$, then $x * y \in A$.
Hence $\mu(x, q) = \mu(y, q) = \mu(x * y, q) = t$ and $\mu(x * y, q) \leq \max\{\mu(x,q), \mu(y,q)\} = t$
If $x,y \notin A$, then $\mu(x,q) = \mu(y,q) = 0$ and $\mu(x * y) \leq \max\{\mu(x),\mu(y)\} = 0$.

If at most one of $x,y \in A$, then at least one of $\mu(x,q)$ and $\mu(y,q)$ is equal to 0.
Therefore, $\max\{\mu(x,q), \mu(y,q)\} = 0$ so that $\mu(x * y) \leq 0$, which completes the proof.

Theorem 5.4

Two lower level KU-sub algebras μ^s, μ^t ($s < t$) of an anti Q-fuzzy sub algebra are equal iff there is no $x \in X$ and $q \in Q$ such that $s \leq \mu(x,q) < t$.

Proof

Let $\mu^s = \mu^t$ for some $s < t$.If there exists $x \in X$ and $q \in Q$ such that $s \leq \mu(x) < t$, then μ^t is a proper subset of μ^s, which is a contradiction.
Conversely, assume that there is no $x \in X$ and $q \in Q$ such that $s \leq \mu(x,q) < t$.
If $x \in \mu^s$, then $\mu(x,q) \leq s$ and $\mu(x,q) \leq t$.
Since $\mu(x,q)$ does not lie between s and t.
Thus $x \in \mu^t$, which gives $\mu^s \subseteq \mu^t$, Also $\mu^t \subseteq \mu^s$.
Therefore $\mu^s = \mu^t$.

Conclusion

In this article we have discussed anti Q-fuzzy KU- ideal of KU-algebras and its lower level cuts in detail.These concepts can further be generalized.

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