An Osculatory Rational Interpolation Method in Linear Systems by Using Routh Model

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Abstract:
In the present paper, theorem for osculatory rational interpolation was shown to establish a new criterion of interpolation and Routh model reduction method for linear systems was explained. On the basis of this conclusion a practical algorithm was presented to get a reduction model of the linear systems. A well-known numerical example of the proposed method is presented to illustrate the correctness effectively.

Keywords: *Osculatory rational interpolation, criterion of interpolation, Routh table, model reduction.*

I.INTRODUCTION

The transfer function of frequency-domain in a linear system is an expression method that describes the system. In practical application, realistic models are so high in dimension that a direct simulation or design would be neither computationally desirable nor physically possible in many cases. Hence, it is significant to reduce the dimension of the system (i.e., the order of transfer function) on the basis of retaining the information of the original model of a large extent. In recent decades, many methods [1-5]had been introduced in scalar and interval systems. The Pade approximation [6,7]was a classical method among them, which made use of the series expansion at zero of the original transfer function G(s), namely it only used the information of each order derivative with G(s) at zero. In the paper the Pade approximation method will be extended to osculatory rational interpolation. The denominator polynomial of the reduced order model is obtained from Routh table and the numerator polynomial is obtained from theinterpolation method. This Pade Approximation method can reduce the original transfer function by taking advantage of the information of each order derivative with the transfer function G(s) at many other points.

This paper has six sections, in section II, the basic concept of Routh model reduction method is explained. In section III, theorem and algorithm of interpolation method is discussed. In section IV, gives the further discussions of the present paper. In section V, numerical example of the proposed method is illustrated. Section VI gives the conclusion.

II.ROUTH MODEL REDUCTION METHOD

Let the transfer function of a higher order system be represented by [6], [7]

$$G(s)=\frac{D_0+D_1 s+\ldots+D_{k-1}s^{k-1}}{e_0+e_1 s+\ldots+e_k s^k}=\frac{D(s)}{E(s)}, \quad (1)$$

Where D_i, i=0,1,…, k-1 are constant l x r matrices, and e_i, i=0,1,…, k are scalar constants.

Assume that the reduced model R (s) of order n is required, and let it be of the form

$$R(s)=\frac{D_n(s)}{E_n(s)}=\frac{A_0+A_1 s+\ldots+A_{n-1}s^{n-1}}{b_0+b_1 s+\ldots+b_{n-1}s^{n-1}+b_n s^n}, \quad (2)$$

where the A_i, i=0,1,…, n -1 are constant l x r matrices, and b_i, i=0,1,…., n are scalar constants.

Algorithm 1

Step 1:The denominator E_n (s) of reduced model transfer function can be constructed from the Routh Stability array of the denominator of the system transfer function as follows.

The Routh stability array is formed by the following

$$b_{i,j}=b_{i-2,j+1}-\frac{b_{i-2,1}b_{i-1,j+1}}{b_{i-1,1}}, \quad (3)$$

$$where\; i\geq 3\; and\; 1\leq j\leq \left[\frac{(k-i+3)}{2}\right]$$

The routh table for the denominator of the system transfer function is given as

$$\begin{array}{lllll} b_{11}=e_k & b_{12}=e_{k-2} & b_{13}=e_{k-4} & b_{14}=e_{k-6} & \dots \\ b_{21}=e_{k-1} & b_{22}=e_{k-3} & b_{23}=e_{k-5} & b_{24}=e_{k-7} & \dots \\ b_{31} & b_{32} & b_{33} & \dots\dots & \\ \dots\dots & & & & \\ b_{k-1,1} & b_{k-1,2} & & & \\ b_{k,1} & & & & \\ b_{k+1,1} & & & & \end{array} \quad (4)$$

E_n (s) may be easily constructed from the (k+1-n) th and (k+2-n) th rows of the above to give

$$E_n(s)=\sum_{j=0}^{n} b_j s^j$$
$$=b_{k+1-n.1}s^n + b_{k+2-n.1}s^{n-1} + b_{k+1-n.2}s^{n-2} + \dots \quad (5)$$

III.THEOREM AND ALGORITHMOF INTERPOLATION METHOD

In this section we first prove a primary theorem and then give a useful algorithm to reduce the linear system models.

Theorem 1 Let $p/q, \hat{p}/\hat{q}$ be two rational fractions. And let $q(s_i)\neq 0, \hat{q}(s_i)\neq 0.$ then

$$\frac{d^{(k)}}{ds^{(k)}}\left(\frac{p}{q}\right)\Big|_{s_i} = \frac{d^{(k)}}{ds^{(k)}}\left(\frac{\hat{p}}{\hat{q}}\right)\Big|_{s_i}$$
$$i=0.1,\dots\dots n, \quad k=0.1,\dots\dots, m_i-1.$$

If and only if there exists a polynomial f(s), such that

$$p\hat{q}-q\hat{p}$$
$$=[(s-s_0)^{m_0}(s-s_1)^{m_1}\dots.(s-s_n)^{m_n}]f(s)$$
$$=g(s)f(s),$$

Where

$g(s) = (s-s_0)^{m_0}(s-s_1)^{m_1}\dots(s-s_n)^{m_n}$

herep,q and $\hat{p},\hat{q}$ are polynomials.

Proof Let

$$p\hat{q}-q\hat{p}=g(s)f(s).$$

Divide on the two sides of the above relation by $q\hat{q}$ to get.

$$\frac{p}{q}-\frac{\hat{p}}{\hat{q}}=\frac{g(s)f(s)}{q\hat{q}}=g(s)\frac{f(s)}{q\hat{q}}$$

Thus

$$\frac{d^{(k)}}{ds^{(k)}}\left(\frac{p}{q}\right)\Big|_{s_i} = \frac{d^{(k)}}{ds^{(k)}}\left(\frac{\hat{p}}{\hat{q}}\right)\Big|_{s_i}$$
$$i=0,1,\dots.,n, \quad k=0,1,\dots.,m_i-1$$

In the following proof we use the mathematical induction. For k=0, obviously,

$$\frac{p(s_i)}{q(s_i)}=\frac{\hat{p}(s_i)}{\hat{q}(s_i)},$$

and it follows that

$$p(s_i)\hat{q}(s_i)-\hat{p}(s_i)q(s_i)=0.$$

Hence

$$p\hat{q}-\hat{p}q=(s-s_i)f_0(s).$$

Where $f_0(s)$ is a polynomial.

Assume that it is true for k≤n-1, that is,

$$\frac{d^{(k)}}{ds^{(k)}}\left(\frac{p}{q}\right)\Big|_{s_i} = \frac{d^{(k)}}{ds^{(k)}}\left(\frac{p}{q}\right)\Big|_{s_i}$$
$$k=0,1,\dots.,n-1. \quad (6)$$

Then $p\hat{q}-\hat{p}q=(s-s_i)^n f_0(s).$

Let a linear time-invariant system in the state-space form [4] be

$$\begin{cases} \dot{x}(t)=Ax(t)+Bu(t). \\ y(t)=Cx(t). \end{cases}$$

Where x, u and y are n-, m-, and r-dimensional state, control, and output vectors, respectively. Using the Laplace transformation, the above system can be represented in the frequency domain as

$$\begin{cases} Y(s)=G(s)U(s), \\ G(s)=C(sI-A)^{-1}B, \end{cases}$$

Where G(s) is the (r x m) - dimensional transfer function matrix whose elements are rational functions of s, i.e.,

$$G(s)=\frac{p(s)}{q(s)}=\frac{b_{n-1}S^{n-1}+...+b_0}{a_n s^n+....+a_0} \quad (7)$$

Which is a transfer function of the original system. Now seek its reduction model

$$\hat{G}(s)=\frac{\hat{p}(s)}{\hat{q}(s)}=\frac{\hat{b}_{m-1}s^{m-1}+....+\hat{b}_0}{\hat{a}_m s^m+....+\hat{a}_0} \quad (8)$$

$$m \le n-1$$

that satisfies the following conditions

$$\hat{G}^{(k)}(s_i)=G^{(k)}(s_i),$$

$$i=0,1,..., j,\ \ k=0,1,...m_i-1,\ \sum_{i=0}^{j} m_i=2m$$

In terms of Theorem 1, it is equivalent to

$$(p\hat{q})^{(k)}\Big|_{s_i}=(q\hat{p})^{(k)}\Big|_{s_i},$$

$$i=0,1,..., j,\ \ k=0,1,...,m_i-1.$$

Now take the following steps respectively:

(1) Divided $p\hat{q}$ by $g(s)=(s-s_0)^{m_0}(s-s_1)^{m_1}\cdots(s-s_j)^{m_j}$ to get the quotient e(s) and the remainder f(s).

(2) Divided $p\hat{q}$ by $g(s)=(s-s_0)^{m_0}(s-s_1)^{m_1}\cdots(s-s_j)^{m_j}$ to get the quotient $l(s)$ and the remainder h(s). It is held that

$$p\hat{q}=g(s)e(s)+f(s), \quad (9)$$

$$q\hat{p}=g(s)l(s)+h(s), \quad (10)$$

Where both f(s) and h(s) are polynomials of degree at most (2m-1).

Thus

$$(p\hat{q})^{(k)}\Big|_{s_i}=f^{(k)}\Big|_{s_i}=(q\hat{p})^{(k)}\Big|_{s_i}=h^{(k)}\Big|_{s_i}$$

$$i=0,1,..., j,\quad k=0,1,...,m_i-1.\sum_{i=0}^{j} m_i=2m$$

By using the basic theorem of algebra, it is obtained that

$$f(s)\equiv h(s). \quad (11)$$

It is found that the coefficient of each term in f(s) in (9) is the linear combination of $\hat{a}_0, \hat{a}_1,...,\hat{a}_m$ and the coefficient of each term in h(s) in (10) is the linear combination of $\hat{b}_0, \hat{b}_1..., \hat{b}_{m-1}$.

By means of the relation (11), a linear system with 2m+1 unknowns and 2m equations is formed. Because the coefficients of a rational fraction as in (8) have one dependent variable, without losing generality, it can be assumed $\hat{a}_0=1$. if the coefficient matrix of the above linear system is nonsingular, then its solution $\hat{b}_0, \hat{b}_1..., \hat{b}_{m-1}, \hat{a}_1..., \hat{a}_m$ can be uniquely determined by using the Cramer rule.

On the basis of the above discussion an algorithm to obtain the reduction model (8) will be presented as follows.

Algorithm 2: Seek the coefficients of the reduction model.

Step 1 Choose 2m points $s_0, s_1, \ldots, s_{2m-1}, s_i \in C$ (they can be multiple) and satisfy $G(s_i) \neq 0$, then compute

$$g(s)=(s-s_0)(s-s_1)....(s-s_{2m-1})$$
$$=(s-s_0)^{m_0}(s-s_1)^{m_1}....(s-s_j)^{m_j}$$
$$=s^{2m}+g_{2m-1}s^{2m-1}+....+g_1 s+g_0$$

Step 2: Compute $p\hat{q}$ and $q\hat{p}$, respectively

$$p\hat{q} = (b_{n-1}s^{n-1} + ... + b_0)(\hat{a}_m s^m + ... + \hat{a}_0)$$
$$= c^{(0)}_{m+n-1}s^{m+n-1} + c^{(0)}_{m+n-2}s^{m+n-2} + ...$$
$$+ c^{(0)}_1 s + c^{(0)}_0,$$
$$c^{(0)}_0 = \hat{a}_0 b_0,$$
$$c^{(0)}_1 = \hat{a}_0 b_1 + \hat{a}_1 b_0,$$
....
$$c^{(0)}_{m+n-2} = \hat{a}_m b_{n-2} + \hat{a}_{m-1} b_{n-1},$$
$$c^{(0)}_{m+n-1} = \hat{a}_m b_{n-1},$$
and
$$q\hat{p} = (a_n s^n + ... + a_0)(\hat{b}_{m-1}s^{m-1} + ... + \hat{b}_0)$$
$$= d^{(0)}_{m+n-1}s^{m+n-1} + d^{(0)}_{m+n-2}s^{m+n-2} + ...$$
$$+ d^{(0)}_1 s + d^{(0)}_0,$$
$$d^{(0)}_0 = \hat{b}_0 a_0,$$
$$d^{(0)}_1 = \hat{b}_0 a_1 + \hat{b}_1 a_0,$$
....
$$d^{(0)}_{m+n-2} = \hat{b}_{m-1} a_{n-1} + \hat{b}_{m-2} a_n,$$
$$d^{(0)}_{m+n-1} = \hat{b}_{m-1} a_n.$$

Step 3: (1) Divide by g(s) to get f(s):

$$\begin{array}{r|l} & c^{(0)}_{m+n-1}s^{n-m-1} + c^{(1)}_{m+n-2}s^{n-m-2} + ... \\ \hline s^{2m} + g_{2m-1}s^{2m-1} + ... + g_1 s + g_0 & c^{(0)}_{m+n-1}s^{m+n-1} + c^{(0)}_{m+n-2}s^{m+n-2} + ... + c^{(0)}_{n-m-1}s^{n-m-1} + ... + c^{(0)}_0 \\ & \underline{c^{(0)}_{m+n-1}s^{m+n-1} + g_{2m-1}c^{(0)}_{m+n-1}s^{m+n-2} + ... + g_0 c^{(0)}_{m+n-1}s^{n-m-1}} \\ & c^{(1)}_{m+n-2}s^{m+n-2} + ... + c^{(1)}_{n-m-2}s^{n-m-3} + ... + c^{(1)}_0 \\ & \underline{c^{(1)}_{m+n-2}s^{m+n-2} + ... + g_0 c^{(1)}_{m+n-2}s^{n-m-2}} \\ & c^{(2)}_{m+n-3}s^{m+n-3} + ... + c^{(2)}_0 \\ & \underline{... \quad ... \quad ... \quad ...} \\ & c^{(n-m)}_{2m-1}s^{2m-1} + ... + c^{(n-m)}_0 \end{array}$$

Thus get the recursive relations:

$$c^{(1)}_i = c^{(0)}_i - c^{(0)}_{m+n-1}g_{i+m-n+1},$$
$$i = 0,1,..., m+n-2,$$
$$c^{(2)}_i = c^{(1)}_i - c^{(1)}_{m+n-2}g_{i+m-n+2},$$
$$i = 0,1,..., m+n-3,$$
$$c^{(3)}_i = c^{(2)}_i - c^{(2)}_{m+n-3}g_{i+m-n+3},$$
$$i = 0,1,..., m+n-4,$$
....
$$c^{(l)}_i = c^{(l-1)}_i - c^{(l-1)}_{m+n-l}g_{i+m-n+l},$$
i=0,1,…, m+n- l -1,

…..

$$c^{(n-m)}_i = c^{(n-m-1)}_i - c^{(n-m-1)}_{2m}g_i,$$
$$i = 0,1,..., 2m-1,$$

When k <0, let g_k =0. In the above the recursive relations, the superscript n in $c^{(n)}_i$ represent thecoefficientswhich are obtained after carrying out the algorithm n steps, and the subscript i in $c^{(n)}_i$ represents the corresponding degree about the variables.

(2)Divide $q\hat{p}$ *by* $g(s)$ *to get* $h(s)$.

Step 4 According to $f(s) \equiv h(s)$, get a linear system with (2m+1) unknowns and 2m equations. let $\hat{a}_0 = 1$ *and solve* $\hat{b}_0,...,\hat{b}_{m-1}, \hat{a}_1,..., \hat{a}_m$ by using the Cramer rule.

IV.FURTHER DISCUSSION

(1) Stabilization

The above method can ensure the reduction model stable. In order to produce the dominator polynomial, the following famous method: the retaining dominant poles will be introduced. Then its numerator can be produced by using the above method, and at this time, the number of points is $\partial(\hat{p}) + 1$.

(2) Choice of the interpolation points

By the way, the above method is similar to the classicalPade approximation when all the interpolation points are zeros, it only takes advantage of information of G(s) at zero.

Usually, interpolation points chosen had better reflect the features of the original model G(s) well. According to experience, we can choose the points which are located in the disk centered at the origin with radius: the distance between origin and the furthest poles; or besides some dominant poles; or besides origin.

V. NUMERICAL EXAMPLE

In this section we give one numerical example to illustrate algorithm 1

Example

For the interpolation points 0, 0.6,2, 3, and 6. Seek a reduction model of type [1/2] for

$$G(s)=\frac{s^4+13s^3+63s^2+133s+102}{s^6+14.5s^5+81s^4+223s^3+318s^2+212.5s+50}$$

Solution:

Routh method applying to the denominator

$$E(s)=s^6+14.5s^5+81s^4+223s^3+318s^2+212.5s+50$$

Routh Table

s^6	1	81	318	50
s^5	14.5	223	212.5	
s^4	65.62	303.3	50	
s^3	155.9	201.2		
s^2	218.6	50		
s^1	165.5			
s^0	50			

Thus the reduced order denominator is

$$E_2(s)=218.6s^2+165.5s+50$$

Using algorithm 1 to get the numerator of reduced order model

$$g(s)=s(s-0.6)(s-2)(s-3)(s-6)$$
$$=s(s^2-9s+18)(s^2-2.6s+1.2)$$
$$g(s)=s^5-11.6s^4+42.6s^3-57.6s^2+21.6s$$

$$p\hat{q}=(s^4+13s^3+63s^2+133s+102)(a_2s^2+a_1s+a_0)$$

$$p\hat{q}=a_2s^6+(133a_2+a_1)s^5+(63a_2+133a_1+a_0)s^4+(133a_2+63a_1+133a_0)s^3$$
$$+(102a_2+133a_1+63a_0)s^2+(102a_1+133a_0)s+102a_0$$
$$p\hat{q}=c_6^0s^6+c_5^0s^5+c_4^0s^4+c_3^0s^3+c_2^0s^2+c_1^0s+c_0^0$$
$$p\hat{q}=g(s)e(s)+f(s)$$

$$f(s)=p\hat{q}-g(s)e(s)$$

$$f(s)=c_5^1s^5+c_4^1s^4+c_3^1s^3+c_2^1s^2+c_1^1s+c_0^1$$

$$f(s)=(133a_2+a_1)s^5+(74.6a_2+133a_1+a_0)s^4$$
$$+(90.4a_2+63a_1+133a_0)s^3+(159.6a_2+133a_1+63a_0)s^2$$
$$+(102a_1+133a_0-21.6a_2)s+102a_0$$
$$q\hat{p}=(s^6+14.5s^5+81s^4+223s^3+318s^2+212.5s+50)(b_1s+b_0)$$
$$q\hat{p}=b_1s^7+(14.5b_1+b_0)s^6+(81b_1+14.5b_0)s^5+(223b_1+81b_0)s^4$$
$$+(318b_1+223b_0)s^3+(212.5b_1+318b_0)s^2$$
$$+(50b_1+21.5b_0)s+50b_0$$
$$q\hat{p}=d_7^0s^7+d_6^0s^6+d_5^0s^5+d_4^0s^4+d_3^0s^3+d_2^0s^2+d_1^0s+d_0^0$$
$$q\hat{p}=g(s)l(s)+h(s)$$
$$h(s)=q\hat{p}-g(s)l(s)$$
$$h(s)=d_6^1s^6+d_5^1s^5+d_4^1s^4+d_3^1s^3+d_2^1s^2+d_1^1s+d_0^1$$
$$h(s)=(14.5b_1+b_0)s^6+(14.5b_0+80b_1)s^5+(81b_0+234b_1)s^4$$
$$+(223b_0+275.4b_1)s^3+(318b_0+270.1b_1)s^2$$
$$+(212.5b_0+28.4b_1)s+50b_0$$

According to $f(s)\equiv h(s)$ the linear system

$$\begin{bmatrix}0 & 0 & 50 & 0\\ -102 & 21.6 & 212.5 & 28.4\\ -133 & -159.6 & 318 & 270.1\\ -63 & -90.4 & 223 & 275.4\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ b_0\\ b_1\end{bmatrix}=\begin{bmatrix}102\\ 133\\ 63\\ 13\end{bmatrix}$$

Is formed

Solve the systems to obtain the reduction model (see fig.1)

$$R_2(s)=\frac{0.665s+2.04}{218.6s^2+165.5s+50}$$

The below figure1 shows the simulation result of comparison of step response for original and reduced order transfer functions.

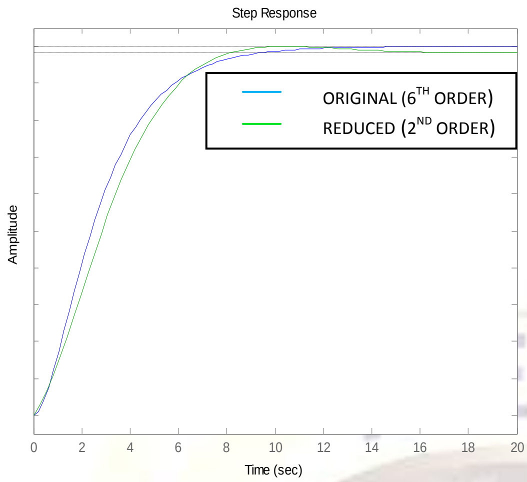

Fig1. Comparison of step response of original and reduced order systems.

VI.CONCLUSION

In this paper, we have observed osculatory rational interpolation to establish a new criterion of interpolation. And the Routh model reduction was introduced to obtain the denominator polynomial of the reduced order transfer function. This Routh method ensures the stability. This method is simple and can be applied to practical control engineering.

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