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Generalized Thermoelastic Problem Concerning Semi–Infinite Rods Problem Of Step In Strain

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Abstract

In this paper, we have solved a generalized thermoelasticity problem concerning to semi – infinite thin rods subjected to step in strain. We obtained the solutions for small values of time.
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Key Words: Generalized thermoelasticity, Temperature distribution, Strain – Acceleration, Relaxation Constant, Stress-Strain relations.

1. Introduction

Transient heat transfer problems usually involve the solution of the classical Fourier heat Conduction equation, which is of parabolic character, as a consequence a perturbed heat signal propagates with an infinite velocity through the medium. That is, if an isotropic homogeneous elastic continuum is subjected to a mechanical or thermal disturbance, the effect of the disturbance will be felt instantaneously at distances infinitely far from its source. Such a behaviour is physically inadmissible and contradicts the existing theories of heat transport mechanisms.

Nonconventional thermoelasticity theories in which the parabolic heat transport equation is replaced by a hyperbolic heat transport equation do not suffer from the above said drawbacks and they admit Wave – like thermal signals propagating with finite speeds. A wave like thermal signal is referred to as second Sound – the first sound being the usual sound wave. Thermoelasticity theories admitting such signals are known as thermoelasticity theories with Second Sound or Generalized thermoelasticity theories or hyperbolic thermoelasticity. A bibliographical review of the literature on the above theory was given by Chandrasekharaiah, D.S [2] in his review article, and "Thermoelasticity with second sound".

Tisza, L [4] predicted the possibility of extremely small heat propagation rates (Second Sound) in liquid Helium – II. Chester, M [5] discussed the possibility of existence of Second Sound in solids. The experiments on Sodium Helium by Ackerman, C.C et.al [1] and by Mc Nelly, T et.al [7] on Sodium fluoride, have shown that second sound occurs in solids also. The second sound effect indicates that heat can be transported by wave type mechanism rather than usual diffusion process. All these researches lead to the reformulation of the existing Fourier heat conduction equation in to a damped wave type equation, which is hyperbolic.

Morse, P.M and Feshbach, H [6] postulated that the governing transient heat conduction must depend upon the velocity of the propagation of heat C. They assumed that the equation,

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{h}\frac{\partial T}{\partial t}=\nabla^2 T$$

Which is hyperbolic, must be the correct governing differential equation for heat conduction problems.

Here, we have studied the application of the generalized theory of thermoelasticity to semi–infinite thin rods when the rods are subjected to Step in strain. Since the specimen is very thin and long the problem is treated as one dimensional.

The solutions of problems of this type usually do not permit closed forms unless certain relaxations in the constraints are made. In order to obtain a closed form solution Lord, H.W. and Schulman Y [3] neglected the strain – acceleration term in the field equations, on the assumption that, for most materials, the relaxation constant and coupled parameter have very small values (< < 1) and that their product has naturally very insignificant value.

Since the problems of this type are amenable to integral transform methods, Laplace transform is used and the solutions are obtained for small values of time.

2. Formulation of the problem:

Consider a long thin rod in which the only non zero stress component is the axial one σ_1. The equation of motion reduces to

$$\frac{\partial \sigma_1}{\partial x_1} = \rho_0 \frac{\partial^2 u_1}{\partial t^2} \qquad (1)$$

Where u_1 is the displacement in the axial direction x_1, ρ_0 is the undeformed density and t is the time. The energy equation of isotropic linear thermoelasticity is given by,

$$k \frac{\partial^2 T}{\partial x_1^2} = \rho_0 C_E \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) + \left(3\lambda + 2\mu\right) \alpha T_0 \left(\frac{\partial e}{\partial t} + \tau_0 \frac{\partial^2 e}{\partial t^2} \right) \qquad (2)$$

The constitutive equation for the isotropic linear elastic solid can be written as

$$(3\lambda + 2\mu)e = \sigma_1 + 3\alpha(3\lambda + 2\mu)T \qquad (3)$$

For the case of thin rod

$$E \frac{\partial u_1}{\partial x_1} = \sigma_1 + \alpha E T \qquad (4)$$

From (2) and (3), we get

$$k \frac{\partial^2 T}{\partial x_1^2} = \rho_0 C_\sigma \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) + \alpha T_0 \left(\frac{\partial \sigma_1}{\partial t} + \tau_0 \frac{\partial^2 \sigma_1}{\partial t^2} \right) \qquad (5)$$

Where,

$$C_\sigma = C_E + \frac{3\alpha^2(3\lambda + 2\mu)T_0}{\rho_0}$$

Here,

$$e = \frac{\partial u_1}{\partial x_1} \qquad (6)$$

From equations (1) and (4), we get

$$\alpha E \frac{\partial^2 T}{\partial x_1^2} = E \frac{\partial^3 u_1}{\partial x_1^3} - \rho_0 \frac{\partial^3 u_1}{\partial t^2 \partial x_1} \qquad (7)$$

Using the following non dimensional variables,

$$x = \frac{x_1}{a}, \quad u = \frac{u_1}{a}, \quad \tau = \frac{vt}{a}, \quad v = \left(\frac{E}{\rho_0}\right)^{1/2}, \quad a = \frac{k}{\rho_0 C_\sigma v}, \quad \theta = \frac{(\rho_o C_\sigma T)}{(\alpha T_o E)}, \quad \sigma = \frac{\sigma_1}{E}$$

$$\beta = \text{Relaxation constant} = \frac{\rho_o C_\sigma \tau_o v^2}{k}$$

$$\delta = \text{Coupling Constant} = \frac{\alpha^2 T_o E}{\rho_o C_\sigma} \qquad (8)$$

We get the field equations as,

$$u^{IV} - (1+\beta)\ddot{u}^{II} - \dot{u}^{II} + (1-\beta)\ddot{u} + \beta(1-\delta)\overset{....}{u} = 0 \qquad (9)$$

and similar equations to σ and θ also. Here, primes denote differentiation with respect to x and superposed dots denote differentiation with respect to time τ. The following auxiliary relations can be obtained from the foregoing equations.

From equation (7), we get

$$\delta\theta^{I} = u^{II} - \ddot{u} \qquad (10)$$

From equation (4), we get

$$\sigma = u^{I} - \delta\theta \tag{10a}$$

From the above two equations, we get

$$\delta\ddot{\theta} = \sigma^{II} - \ddot{\sigma} \tag{11}$$

From equation (5), we get

$$\theta^{II} - \dot{\theta} - \beta\ddot{\theta} = \dot{\sigma} + \beta\ddot{\sigma} \tag{12}$$

From equations (10), (10 a) and (12), we get,

$$\delta(1-\delta)\dot{\theta} + \beta\delta(1-\delta)\ddot{\theta} = u^{III} - (1+\beta\delta)\ddot{u}^{I} - \delta\dot{u}^{I} \tag{13}$$

Applying Laplace transform to equation (9), we get

$$\bar{u}^{IV} - [(1+\beta)p^{2} + p]\bar{u}^{II} + (1-\delta)p^{3}(1+\beta p)\bar{u} = 0 \tag{14}$$

Where, $\bar{u}(x, p) = \int_{o}^{\infty} u(x, \tau).e^{(-p\tau)d\tau}$

while applying Laplace transform, we have used the initial conditions

$$u(x,0) = \dot{u}(x,0) = \ddot{u}(x,0) = \dddot{u}(x,0) = 0$$

If λ_1^2, λ_2^2 are the roots of the characteristic equation of (14), we get

$$\lambda_1^{2} + \lambda_2^{2} = p[(1+\beta)p + 1]$$

$$\lambda_1^{2} - \lambda_2^{2} = p[(1+\beta)p+1]^{2} - 4p(1-\delta)(1+\beta p)]^{1/2} \tag{15}$$

Solving, we get

$$\lambda_{1,2} = \left[\frac{p}{2}\left[(1+\beta)p + 1 \pm \left[((1+\beta)p+1)^{2} - 4p(1-\delta)(1+\beta p)\right]^{1/2}\right]\right]^{1/2} \tag{16}$$

For large values of p, we get

$$\lambda_{1,2} = \frac{p}{V_{1,2}} + k_{1,2} + O\left(\frac{1}{p}\right) \tag{17}$$

Where, $\frac{2}{v_{1,2}^2} = 1 + \beta \pm \Gamma^{1/2}$

$$4K_{1,2} = V_{1,2}[1 \mp (1-\beta-2\delta)\Gamma^{-1/2}] \quad \text{and} \quad \Gamma = (1-\beta)^2 + 4\beta\delta$$

As we see $v_1 < v_2$, we get

$$\frac{1}{\lambda_1^2 - \lambda_2^2} = p^{-3}\Gamma^{-3/2}[p\Gamma + 1 - \beta - 2\delta] + O\left(\frac{1}{p}\right)$$

For small values of p, we get $\lambda_1 = p^{1/2}$, $\lambda_2 = 0$ (18)

3. Problem of step in strain

The boundary and initial conditions are,

$$\in(0,\tau) = \left.\frac{\partial u}{\partial x}\right|_{x=0} = \begin{cases} 0 & for \quad \tau < 0 \\ \in_0 & for \quad \tau > 0 \end{cases} \tag{19}$$

$$\theta(0,\tau) = 0$$

We know the governing equation satisfied by displacement u is (equation 9)

$$u^{IV} - (1+\beta)\ddot{u}^{II} - \dot{u}^{II} + (1-\delta)\dddot{u} + \beta(1-\delta)\ddddot{u} = 0$$

Proceeding as in previous problems, applying Laplace transform and using boundary conditions (19), we get

$$\frac{\bar{u}^{1}}{\epsilon_0} = \frac{1}{p(\lambda_2^2 - \lambda_1^2)}[(1+\beta\delta)p^2 + \delta p - \lambda_1^2]e^{(-\lambda_2 x)}$$
$$-[(1+\beta\delta)p^2 + \delta p - \lambda_2^2]e^{(-\lambda_1 x)} \qquad (20)$$

Taking inverse transform for large values of p, we get

$$\frac{u^I}{\epsilon_0} = \begin{cases} 0 & for \quad \tau < \frac{x}{v_2} \\ \frac{-e^{(-k_2 x)}}{\Gamma^{\frac{1}{2}}}\left[\left(1+\beta\delta - \frac{1}{2v1}\right) + \left[\delta - \frac{2k_1}{v_1} + \frac{1-\beta-2\delta}{\Gamma}\left(1+\beta\delta-\frac{1}{2v1}\right)\right]\left(\tau - \frac{x}{v_1}\right)\right] & for \quad \frac{x}{v_2} < \tau < \frac{x}{v_1} \\ \frac{e^{(-k_1 x)}}{\Gamma^{\frac{1}{2}}}\left[\left(1+\beta\delta - \frac{1}{2v2}\right) + \left[\delta - \frac{2k_2}{v_2} + \frac{1-\beta-2\delta}{\Gamma}\left(1+\beta\delta-\frac{1}{2v2}\right)\right]\left(\tau - \frac{x}{v_1}\right)\right] \\ -\frac{-e^{(-k_2 x)}}{\Gamma^{\frac{1}{2}}}\left[\left(1+\beta\delta - \frac{1}{2v1}\right) + \left[\delta - \frac{2k_1}{v_1} + \frac{1-\beta-2\delta}{\Gamma}\left(1+\beta\delta-\frac{1}{2v1}\right)\right]\left(\tau - \frac{x}{v_2}\right)\right] & for \quad \tau > \frac{x}{v_1} \end{cases} \qquad (21)$$

From equation (13), the temperature distribution is given by,

$$
\frac{\theta}{\epsilon_0}=\begin{cases}
-\dfrac{1}{\delta}[(1+\beta\delta)v_1^2v_2^2+1-v_1^2-v_2^2] & \text{for } \tau<\dfrac{x}{v_2}\\[2ex]
\dfrac{-e^{(-k_2x)}}{\delta\Gamma^{\frac{1}{2}}v_1^2}\left[\left(1-v_2^2\right)\left(v_1^2+v_1^2\delta\beta-1\right)+\left[\begin{aligned}&2k_2v_2\left(1-v_1^2-v_1^2\beta\delta\right)\left(1-v_2^2\right)\\&-\left(1-v_2^2\right)\left(2k_1v_1+v_1^2\delta\right)\\&+\frac{1-\beta-2\delta}{\Gamma}\left(1-v_2^2\right)\left(v_1^2+v_1^2\delta\beta-1\right)\end{aligned}\right]\left(\tau-\frac{x}{v_2}\right)\right] \\
\quad-\dfrac{\left(\left(1+\beta\delta\right)v_1^2v_2^2+1-v_1^2-v_2^2\right)}{\delta} & \text{for } \dfrac{x}{v_2}<\tau<\dfrac{x}{v_1}\\[2ex]
\dfrac{e^{(-k_1x)}}{\delta\Gamma^{\frac{1}{2}}v_2^2}\left[\left(1-v_1^2\right)\left(v_2^2+v_2^2\delta\beta-1\right)+\left[\begin{aligned}&2k_1v_1\left(1-v_2^2-v_2^2\beta\delta\right)\left(1-v_1^2\right)\\&-\left(1-v_1^2\right)\left(2k_2v_2+v_2^2\delta\right)\\&+\frac{1-\beta-2\delta}{\Gamma}\left(1-v_1^2\right)\left(v_2^2+v_2^2\delta\beta-1\right)\end{aligned}\right]\left(\tau-\frac{x}{v_1}\right)\right] \\
\dfrac{-e^{(-k_2x)}}{\delta\Gamma^{\frac{1}{2}}v_1^2}\left[\left(1-v_2^2\right)\left(v_1^2+v_1^2\delta\beta-1\right)+\left[\begin{aligned}&2k_1v_2\left(1-v_1^2-v_1^2\beta\delta\right)\left(1-v_2^2\right)\\&-\left(1-v_2^2\right)\left(2k_2v_1+v_1^2\delta\right)\\&+\frac{1-\beta-2\delta}{\Gamma}\left(1-v_2^2\right)\left(v_1^2+v_1^2\delta\beta-1\right)\end{aligned}\right]\left(\tau-\frac{x}{v_2}\right)\right] \\
\quad-\dfrac{\left(\left(1+\beta\delta\right)v_1^2v_2^2+1-v_1^2-v_2^2\right)}{\delta} & \text{for } \tau>\dfrac{x}{v_1}
\end{cases}
\tag{22}
$$

For most of the materials the parameters β and δ have smaller values so that the product $\beta\delta$ is far less than Unity. Based on this assumption, Lord, H.W. and Shulman, Y [3] neglected the term containing $\beta\delta$, which is the term involving strain acceleration. Here we want to study the effect of dropping this term from the field equations.

Neglecting the term containing $\beta\delta$, we get the field equations as

$$u^{IV} - (1+\beta)\ddot{u}^{II} - \dot{u}^{II} + (1-\delta)\dddot{u} + \beta \ddddot{u} = 0 \tag{23}$$

$$\delta\theta^{I} = u^{II} - \ddot{u} \tag{24}$$

$$\sigma = u^{I} - \delta\theta \tag{25}$$

$$\delta\ddot{\theta} = \sigma^{II} - \ddot{\sigma} \tag{26}$$

$$\theta^{II} - \dot{\theta} - \beta\ddot{\theta} = \dot{\sigma} + \beta\ddot{\sigma} \tag{27}$$

and,

$$\delta\dot{\theta} = u^{III} - \ddot{u}^{I} - \delta\dot{u}^{I} \tag{28}$$

Applying Laplace transform, we get

$$\bar{u}^{IV} - [(1+\beta)p^2 + p]\bar{u}^{II} + [1-\delta+\beta p]p^3\bar{u} = 0 \tag{29}$$

If λ_1 and λ_2 are the characteristic roots of (29), we have

$$\lambda_1^2 + \lambda_2^2 = p[(1+\beta)p+1],$$

$$\lambda_1^2\lambda_2^2 = (1-\delta+\beta p)p^3$$

$$\lambda_1^2 - \lambda_2^2 = p[(1+\beta)p+1]^2 - 4p(1-\delta-\beta p)]^{1/2}$$

$$\therefore \lambda_{1,2} = \left[\frac{p}{2}[(1+\beta)p+1 \pm [(1+\beta)p+1)^2\right] - 4p(1-\delta+\beta p)^{1/2}]^{1/2} \tag{30}$$

For large values of p,

$$\lambda_{1,2} = \frac{p}{V_{1,2}} + K_{1,2} + O\left(\frac{1}{p}\right)$$

Where,

$$\frac{2}{v_{1,2}^{2}} = 1 + \beta \pm \Gamma^{1/2}$$

$$4K_{1,2} = V_{1,2}[1 \mp (1 - \beta - 2\delta)\Gamma^{-1/2}]$$

$$\Gamma = (1 - \beta)^{2} \tag{31}$$

For the problem of velocity impact, from equation (29), taking the regularity boundary condition at infinity into account, we get

$$\bar{u}(x, p) = A\,e^{(-\lambda_1 x)} + \beta\,e^{(-\lambda_2 x)}$$

Here A and B are constants and are determined using boundary conditions.

$$\therefore \frac{\bar{u}^{I}}{v_0} = \frac{1}{p^{2}(\lambda_2^{2} - \lambda_1^{2})}\left[\lambda_2(\lambda_1^{2} - p^{2})\,e^{(-\lambda_2 x)} - \lambda_1(\lambda_2^{2} - p^{2})\,e^{(-\lambda_1 x)}\right]$$

For large values of p, we get

$$\frac{u^I}{v_0}=\begin{cases} 0, & for \quad \tau<\dfrac{x}{v_2} \\[2ex] -\dfrac{v_2 e^{\left(-k_2x\right)}}{v_1^2 v_2^2 \Gamma^{\frac{1}{2}}}\left[\left(1-v_1^2\right)+\left[\dfrac{\left(1-\beta-2\delta\right)\left(1-v_1^2\right)}{\Gamma}+2k_1v_1+k_2v_2\left(1-v_1^2\right)\right]\left(\tau-\dfrac{x}{v_2}\right)\right] & \\[2ex] & for \quad \dfrac{x}{v_2}<\tau<\dfrac{x}{v_1} \\[2ex] \dfrac{1}{v_1^2 v_2^2 \Gamma^{\frac{1}{2}}}\left[\begin{array}{l} v_1 e^{\left(-k_1x\right)}\left[\left(1-v_2^2\right)+\left[2k_2v_2+k_1v_1\left(1-v_2^2\right)\dfrac{\left(1-\beta-2\delta\right)\left(1-v_2^2\right)}{\Gamma}\right]\left(\tau-\dfrac{x}{v_1}\right)\right] \\ -v_2 e^{\left(-k_2x\right)}\left[\left(1-v_1^2\right)+\left[\dfrac{\left(1-\beta-2\delta\right)\left(1-v_1^2\right)}{\Gamma}+2k_1v_1+k_2v_2\left(1-v_1^2\right)\right]\left(\tau-\dfrac{x}{v_2}\right)\right]\end{array}\right] & \\[2ex] & for \quad \tau>\dfrac{x}{v_1} \end{cases} \qquad (32)$$

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