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Optimal Design of Failure Step Stress Partially Accelerated Life Tests with Type II Censored Inverted Weibull Data

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ABSTRACT

This article provides the optimum simple failure step stress partially accelerated life tests (FSS-PALTs) and statistical inferences for the model parameters and acceleration factor in which items are run at both accelerated and use conditions. It is assumed that the lifetime of the test items follows inverse Weibull distribution under type II censoring. The maximum likelihood estimators (MLEs), asymptomatic variance-covariance matrix, and the confidence bounds of the model parameters and acceleration factor are obtained via MathCAD"14". The optimum test a plan specifies the optimal stress switching point is determined by minimizes the generalized asymptotic variance of the MLEs for the model parameters. Finally, the numerical studies are applied to illustrate the proposed procedures.

Keywords - **Failure step stress test, Generalized asymptotic variance, Inverse Weibull distribution.**

I. INTRODUCTION

Traditional life data analysis involves analyzing times to-failure data (of a product, system or component) obtained under normal use conditions in order to determine the life characteristics of the product, system or component. In many situations, and for many reasons, the life data is very difficult, if not impossible, to obtain. For this reason, partially accelerated life test (PALT) is the reasonable procedure to be conducted. PALT allows the experimenter to apply more severe stress to obtain information on the parameters of lifetime distribution more quickly than would be possible under normal operating conditions in short period of time.

As seen partially accelerated life tests are more suitable test to be performed for which units are subjected to both normal and accelerated conditions. According to Nelson [1], the stress can be applied in various ways. One way to accelerate failure is the step-stress, which increases the stress applied to test unit at a specified discrete sequence. Concerning the step-stress test method there are two main types, the first one is time step stress PALT (TSS-PALT) where this test runs for a specified time at each stress. The second one is failure-step stress (FSS-PALT) whereas this test runs until a specified number of failure units at each stress.

This article considers the simple failure step stress PALT, this test runs only under two stresses (use and accelerated) stress. There were no more studies had been done about FSS-PALT, unless Ismail and Aly [2] studied the optimum test plans of FSS-PALT for the Weibull distribution under type-II censored, which determine the optimum number of failure units at used stress to switch to the accelerated stress. The model parameters are estimated using maximum likelihood method. Also, the confidence intervals and the Fisher information matrix of the estimated parameters are obtained. On other side, TSS-PALTs have been studied by several authors for example; DeGroot and Goel [3], Bai and Chung[4], Attia, Abel-Ghaly and Abel-Ghani [5] , Abdel-Ghaly, Attia and Abdel-Ghani [6], Abdel-Ghaly, Attia, and Abdel-Ghani [7] , Abdel-Ghani[8], Abd-Elfattah, Hassan and Nassr [9], Ismail and Sarhan [10] and Ismail [11].

This article concerns with the simple failure-step stress PALT; it is assumed that the lifetimes of test items follow inverse Weibull distribution based on censoring samples. The maximum likelihood approach is applied as estimation procedure under type II censoring. Asymptotic variance-covariance matrix of the estimators and the confidence interval of the unknown parameters and acceleration factor are obtained for large sample sizes. In addition, optimum test plans for simple failure-step stress test are developed.

This article can be organized as follows. In Section 2 the inverse Weibull (IW) distribution and the test procedure in FSS-PALTs are introduced. Section 3 presents the maximum likelihood estimators (MLEs) of model parameters based on type II censored samples, also an approximate asymptotic variances and covariance matrix are investigated. The problem of choosing the optimal test plan under normal stress is addressed in Section 4. Section 5 presents the confidence intervals for the model parameters based on asymptotic variance covariance matrix. Section 6 explains the simulation studies for illustrating the theoretical results. Section 7 shows the simulation results. Finally, conclusions are included in Section 8.

II. MODEL DESCRIPTION AND TEST PROCEDURE

The inverse Weibull distribution was developed by Erto [12], it is used in reliability analysis. It can be successful in modelling life for many devices and variables such as electron tubes, capacitors, generators etc. It has been derived as a suitable model for describing degradation phenomena of mechanical components such as the dynamic components of diesel engines. IW distribution provides a good fit to several data sets such as the times to breakdown of an insulating fluid subject to the action of a constant tension. (Nelson [13])

The IW distribution has the distribution function:

$$F(t,\lambda,\alpha)=e^{-\lambda t^{-\alpha}}; \qquad t,\alpha,\lambda>0. \tag{2.1}$$

Therefore, IW distribution has a probability density function

$$f(t,\lambda,\alpha)=\alpha\lambda t^{-(\alpha+1)}e^{-\lambda t^{-\alpha}}, \tag{2.2}$$

The reliability function

$$R(t,\lambda,\alpha)=1-e^{-\lambda t^{-\alpha}}, \tag{2.3}$$

and the hazard function

$$h(t,\lambda,\alpha)=\frac{\alpha\lambda t^{-(\alpha+1)}}{e^{\lambda t^{-\alpha}}-1}. \tag{2.4}$$

Here α is a shape parameter and λ is scale parameter. Note that, when $\alpha=1$ the distribution is the same as the inverse exponential distribution for a constant hazard. In particular, when $\alpha=2$ it is known as the inverse Rayleigh distribution. When $\alpha<1$, the hazard function is continually decreasing which represents early failures. When $\alpha>1$, the hazard function is continually increasing which represents wear-out failures.

In this Section, FSS-PALT model for inverse Weibull lifetime data under type II censoring is assumed. The test is conducted as follows, a random sample of n independent and identically units firstly tested under normal conditions until time y_{n_1}, where n1 = πn units are failed under normal condition. After time y_{n_1} the unfailed units (n-n1) are subjected at accelerated conditions and continued under these conditions until censoring time y_r is reached, where y_r is the time of failed r units (r= n1+na) which is predetermined, na is the number of failure units under accelerated condition and the number of censoring units are (n-r=nc). The effect of this switch is to multiply the remaining lifetime of units by the inverse of the accelerator factor β. In this case, the switching to the higher stress will shorten the life of test units.

The following assumptions are made:

(1) The failure times $y_i, i=1,\dots,n$ are independent and identically distributed random variables.

(2) The total lifetime of test units denoted by Y pass through two stage, which are the normal and accelerated conditions. Then, the lifetime of the unit under FSS-PALT can be written as

$$Y=\begin{cases} T & if \quad T\le y_{n_1} \\ y_{n_1}+\beta^{-1}(T-y_{n_1}) & if \quad T>y_{n_1} \end{cases} \tag{2.5}$$

This Tampered Random Variable (TRV) model was proposed by DeGroot and Goel [3].

From the assumptions, the probability density function of a total lifetime Y of a unit takes the form:

$$f(y) = \begin{cases} \alpha\lambda\, y^{-(\alpha+1)}\, e^{-\lambda y^{-\alpha}} & 0 < y \le y_{n_1} \\ \alpha\lambda\beta\ \left((y-y_{n_1})\beta + y_{n_1}\right)^{-(\alpha+1)}\ e^{-\lambda\left((y-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}} & y > y_{n_1} \end{cases} \tag{2.6}$$

where $\beta > 1$, $\alpha > 0$ and $\lambda > 0$.

III. MAXIMUM LIKELIHOOD ESTIMATORS

According to type II censoring, the test applied to n identical units will terminate when r units fail at time y_r $(0 < y_{n_1} < y_r < \infty)$. Let n1 and na denote the number of failures that occur before y_{n_1} and the number of failures that occur before y_r at normal and accelerated conditions, respectively. Hence, the observed values of the total lifetime Y are

$$y_1 < \cdots < y_{n_1-1} \le y_{n_1} < y_{n_1+1} < \cdots < y_{n_1+n_a-1} \le y_r,$$

Let U_{1i} and U_{2i} be indicator functions such that:

$$U_{1i} = \begin{cases} 1 & \text{if } \ y_i \le y_{n_1} \\ 0 & \text{otherwise} \end{cases} \qquad i = 1,2,\dots,n$$

and,

$$U_{2i} = \begin{cases} 1 & \text{if } \ y_{n_1} < y_i \le r \\ 0 & \text{otherwise} \end{cases} \qquad i = 1,2,\dots,n$$

Then, the likelihood function under type II censoring can be written as

$$L(\lambda,\alpha,\beta) \propto \prod_{i=1}^{n} \{f_1(y_i)\}^{U_{1i}} \{f_2(y_i)\}^{U_{2i}} \{R(y_r)\}^{\bar{U}_{1i}\bar{U}_{2i}},$$

$$L(\alpha,\lambda,\beta) \propto \prod_{i=1}^{n} \left[\begin{array}{l} \left\{\alpha\lambda\, y_i^{-(\alpha+1)} e^{-\lambda y_i^{-\alpha}}\right\}^{U_{1i}} \\ \left\{\alpha\lambda\beta\left((y_i-y_{n_1})\beta+y_{n_1}\right)^{-(\alpha+1)} e^{-\lambda[\ (y_i-y_{n_1})\beta+y_{n_1}]^{-\alpha}}\right\}^{U_{2i}} \\ \left\{1-e^{-\lambda[\ (y_r-y_{n_1})\beta+y_{n_1}]^{-\alpha}}\right\}^{\bar{U}_{1i}\bar{U}_{2i}} \end{array} \right] \tag{3.1}$$

where, $\bar{U}_{1i} = 1 - U_{1i}$ *and* $\bar{U}_{2i} = 1 - U_{2i}$.
It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of the likelihood function $\ln L \equiv \ln L(\alpha,\lambda,\beta)$ is given by

$$\begin{aligned} \ln L &= (n_1+n_a)\ln\alpha + (n_1+n_a)\ln\lambda \\ &+ n_a\ln\beta - (\alpha+1)\left\{\sum_{i=1}^{n} U_{1i}\ln y_i + \sum_{i=1}^{n} U_{2i}\ln((y_i-y_{n_1})\beta+y_{n_1})\right\} \\ &- \lambda\left\{\sum_{i=1}^{n} U_{1i}\, y_i^{-\alpha} + \sum_{i=1}^{n} U_{2i}((y_i-y_{n_1})\beta+y_{n_1})^{-\alpha}\right\} \\ &+ n_c\ln\left[1-e^{-\lambda((y_r-y_{n_1})\beta+y_{n_1})^{-\alpha}}\right] \end{aligned} \tag{3.2}$$

where,

$$n_1=\sum_{i=1}^{n}U_{1i},\, n_a=\sum_{i=1}^{n}U_{2i},\ n_c=\sum_{i=1}^{n}\overline{U_{1i}}\,\overline{U_{2i}}$$

MLEs $\hat{\lambda}$, $\hat{\alpha}$, and $\hat{\beta}$ of λ, α and β are the solutions of the following system of equations obtained by take the first partial derivatives of the log likelihood function with respect to λ, α and β and equalize it to zero. The system of equations is as follows:

$$\frac{\partial \ln L}{\partial \lambda}=\frac{n_1+n_a}{\hat{\lambda}}-\sum_{i=1}^{n}U_{1i}\,y_i^{-\hat{\alpha}}-\sum_{i=1}^{n}U_{2i}\left((y_i-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}$$
$$+\frac{n_c\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}e^{-\hat{\lambda}[(y_r-y_{n_1})\hat{\beta}+y_{n_1}]^{-\hat{\alpha}}}}{1-e^{-\hat{\lambda}\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}}}=0\ , \tag{3.3}$$

$$\frac{\partial \ln L}{\partial \alpha}=\frac{n_1+n_a}{\hat{\alpha}}-\sum_{i=1}^{n}U_{1i}\ln y_i-\sum_{i=1}^{n}U_{2i}\ln\left((y_i-y_{n_1})\hat{\beta}+y_{n_1}\right)$$
$$+\hat{\lambda}\left[\sum_{i=1}^{n}U_{1i}y_i^{-\hat{\alpha}}\ln y_i+\sum_{i=1}^{n}U_{2i}\left((y_i-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}\ln\left((y_i-y_{n_1})\hat{\beta}+y_{n_1}\right)\right]$$
$$-\frac{n_c\hat{\lambda}\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}\ln\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)e^{-\hat{\lambda}\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}}}{1-e^{-\hat{\lambda}\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}}}=0\ , \tag{3.4}$$

$$\frac{\partial \ln L}{\partial \beta}=\frac{n_a}{\hat{\beta}}-(\hat{\alpha}+1)\sum_{i=1}^{n}\frac{U_{2i}(y_i-y_{n_1})}{\left((y_i-y_{n_1})\hat{\beta}+y_{n_1}\right)}+\hat{\alpha}\hat{\lambda}\sum_{i=1}^{n}\frac{U_{2i}(y_i-y_{n_1})}{\left((y_i-y_{n_1})\hat{\beta}+y_{n_1}\right)^{\hat{\alpha}+1}}$$
$$-\frac{n_c\hat{\alpha}\hat{\lambda}(y_r-y_{n_1})\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}-1}e^{-\hat{\lambda}\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}}}{1-e^{-\hat{\lambda}\left((y_r-y_{n_1})\hat{\beta}+y_{n_1}\right)^{-\hat{\alpha}}}}=0. \tag{3.5}$$

Obviously, it is difficult to obtain a closed form solution to nonlinear equations (3.3),(3.4) and (3.5).

So, Newton-Raphson method is used to solve these equations simultaneously to obtain $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\beta}$ via MathCAD "14".

Note that, $n_1=\pi n$ where π is the proportion of the failure units at normal condition that pre-specified.

The asymptotic variances and covariance matrix of the MLE of the parameters can be approximated by numerically inverting the asymptotic Fisher-information matrix F. It is composed of the negative second and mixed derivatives of the natural logarithm of the likelihood function evaluated at the MLE. The asymptotic Fisher information matrix F can be written as follows:

$$F=\begin{bmatrix}f_{11}&f_{12}&f_{13}\\f_{21}&f_{22}&f_{23}\\f_{31}&f_{32}&f_{33}\end{bmatrix}=\begin{bmatrix}\frac{-\partial^2\ln L}{\partial\lambda^2}&\frac{-\partial^2\ln L}{\partial\lambda\partial\alpha}&\frac{-\partial^2\ln L}{\partial\lambda\partial\beta}\\\frac{-\partial^2\ln L}{\partial\alpha\partial\lambda}&\frac{-\partial^2\ln L}{\partial\alpha^2}&\frac{-\partial^2\ln L}{\partial\alpha\partial\beta}\\\frac{-\partial^2\ln L}{\partial\beta\partial\lambda}&\frac{-\partial^2\ln L}{\partial\beta\partial\alpha}&\frac{-\partial^2\ln L}{\partial\beta^2}\end{bmatrix}\downarrow(\hat{\lambda},\hat{\alpha},\hat{\beta})$$

The second and mixed partial derivatives of the log-likelihood function with respect to the parameters to λ, α and β are obtained as the following

$$f_{11}=\frac{(n_1+n_a)}{\lambda^2}+\frac{n_c\left((y_r-y_{n_1})\beta+y_{n_1})\right)^{-2\alpha}e^{-\lambda((y_r-y_{n_1})\beta+y_{n_1})^{-\alpha}}}{[1-e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1})\right)^{-\alpha}}]^2}.$$

$$f_{22}=\frac{(n_1+n_a)}{\alpha^2}$$
$$+\lambda\left[\sum_{i=1}^{n}U_{1i}y_i^{-\alpha}(\ln y_i)^2+\sum_{i=1}^{n}U_{2i}\left((y_i-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}[\ln\left((y_i-y_{n_1})\beta+y_{n_1}\right)]^2\right]$$
$$+\frac{n_c\lambda\left((y_r-y_{n_1})\beta+y_{n_1})\right)^{-\alpha}[\ln\left((y_r-y_{n_1})\beta+y_{n_1})\right]^2e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}}{[1-e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}]^2}$$
$$\times\left\{\lambda\left((y_r-y_{n_1})\beta+y_{n_1})\right)^{-\alpha}-1+e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}\right\},$$

$$f_{33}=\frac{n_a}{\beta^2}$$
$$-(\alpha+1)\left[\sum_{i=1}^{n}\frac{U_{2i}\left(y_i-y_{n_1}\right)^2}{((y_i-y_{n_1})\beta+y_{n_1})^2}-\alpha\lambda\sum_{i=1}^{n}U_{2i}(y_i-y_{n_1})^2\left((y_i-y_{n_1})\beta+y_{n_1}\right)^{-\alpha-2}\right]$$
$$-\frac{n_c\alpha\lambda(y_r-y_{n_1})^2((y_r-y_{n_1})\beta+y_{n_1})^{-\alpha-2}e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}}{[1-e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}]^2}$$
$$\times\left\{(\alpha+1)\left[1-e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}\right]-\alpha\lambda((y_r-y_{n_1})\beta+y_{n_1})^{-\alpha}\right\}.$$

$$f_{12}=-\sum_{i=1}^{n}U_{1i}y^{-\alpha}\ln y_i-\sum_{i=1}^{n}U_{2i}\left((y_i-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}\ln\left((y_i-y_{n_1})\beta+y_{n_1}\right)$$
$$-\frac{n_c\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}\ln\left((y_r-y_{n_1})\beta+y_{n_1}\right)e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}}{[1-e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}]^2}$$
$$\times\left\{\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}+e^{-\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}}-1\right\}.$$

$$f_{13}=-\alpha\sum_{i=1}^{n}\frac{U_{2i}(y_i-y_{n_1})}{\left((y_i-y_{n_1})\beta+y_{n_1}\right)^{\alpha+1}}$$
$$-\frac{n_c\alpha(y_r-y_{n_1})\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha-1}e^{-\lambda[(y_r-y_{n_1})\beta+y_{n_1}]^{-\alpha}}}{[1-e^{-\lambda((y_r-y_{n_1})\beta+y_{n_1})^{-\alpha}}]^2}$$
$$\times\left\{\lambda\left((y_r-y_{n_1})\beta+y_{n_1}\right)^{-\alpha}+e^{-\lambda[(y_r-y_{n_1})\beta+y_{n_1}]^{-\alpha}}-1\right\}.$$

$$f_{23} = \sum_{i=1}^{n} \frac{U_{2i}(y_i - y_{n_1})}{\left((y_i - y_{n_1})\beta + y_{n_1}\right)}$$

$$-\lambda\left[\sum_{i=1}^{n} \frac{-\alpha U_{2i}(y_i - y_{n_1})\ln\left((y_i - y_{n_1})\beta + y_{n_1}\right)}{\left((y_i - y_{n_1})\beta + y_{n_1}\right)^{\alpha+1}} + \sum_{i=1}^{n} \frac{U_{2i}(y_i - y_{n_1})}{\left((y_i - y_{n_1})\beta + y_{n_1}\right)^{\alpha+1}}\right]$$

$$+\frac{\lambda n_c (y_r - y_{n_1})\left((y_r - y_{n_1})\beta + y_{n_1}\right)^{-\alpha-1} e^{-\lambda[(y_r - y_{n_1})\beta + y_{n_1}]^{-\alpha}}}{[1 - e^{-\lambda((y_r - y_{n_1})\beta + y_{n_1})^{-\alpha}}]^2} \times$$

$$\left\{ \begin{array}{l} \left(\begin{array}{l} 1 - e^{-\lambda[(y_r - y_{n_1})\beta + y_{n_1}]^{-\alpha}} \\ -\lambda\left((y_r - y_{n_1})\beta + y_{n_1}\right)^{-\alpha} \ln\left((y_r - y_{n_1})\beta + y_{n_1}\right) e^{-\lambda[(y_r - y_{n_1})\beta + y_{n_1}]^{-\alpha}} \end{array} \right) \\ - \left(\begin{array}{l} \alpha \ln\left((y_r - y_{n_1})\beta + y_{n_1}\right) \\ \times \left[\begin{array}{l} 1 - \lambda\left((y_r - y_{n_1})\beta + y_{n_1}\right)^{-\alpha} - e^{-\lambda[(y_r - y_{n_1})\beta + y_{n_1}]^{-\alpha}} \\ +\lambda\left((y_r - y_{n_1})\beta + y_{n_1}\right)^{-\alpha} e^{-\lambda[(y_r - y_{n_1})\beta + y_{n_1}]^{-\alpha}} \end{array} \right] \end{array} \right) \end{array} \right\}.$$

IV. Optimum Test Plan

The optimal plans for FSS-PALT consider the optimum proportion of test units that must fail at normal stress according to a certain optimality criterion which is a GAV of the MLE of the model parameters. The GAV for the MLE of the model parameters is the reciprocal of the determination of the Fisher information matrix (Bai, Chung and Chung [14]). That is,

$$GAV\,(\hat{\lambda}, \hat{\alpha}, \hat{\beta}) = \frac{1}{|F|} \tag{4.1}$$

The optimum test plan for products having inverse Weibull lifetime distribution is to find the optimum proportion of test units failing at normal stress π^* such that the GAV is minimized. The minimization of GAV over π can be achieved by solving the following equation:

$$\frac{\partial GAV}{\partial \pi} = 0 \tag{4.2}$$

This is reduced to

$$\frac{\partial |F|}{\partial \pi} = 0 \;, \tag{4.3}$$

where,

$$\begin{aligned} \frac{\partial |F|}{\partial \pi} &= f_{11}(f_{22}f_{33}' + f_{22}'f_{33} - 2f_{23}f_{23}') + f_{11}'(f_{22}f_{33} - f_{23}^2) \\ &\quad - f_{12}(f_{12}f_{33}' + f_{12}'f_{33} - f_{23}f_{13}' - f_{23}'f_{13}) - f_{12}'(f_{12}f_{33} - f_{23}f_{13}) \\ &\quad + f_{13}(f_{12}f_{23}' + f_{12}'f_{23} - f_{22}f_{13}' - f_{22}'f_{13}) + f_{13}'(f_{12}f_{23} - f_{22}f_{13}). \end{aligned} \tag{4.4}$$

In general, the solution to equation (4.4) is not in a closed form. So, Newton-Raphson method is applied to obtain the optimal stress change point π^* which minimizing the GAV.

Thus, the optimal numbers of units must fail at normal use condition ($n_1{}^*$) for switching to accelerated condition is:

$$n_1^* = \pi^* n \tag{4.5}$$

where n is the sample size.

V. Confidence Interval of Model Parameters

For large sample size n, the MLEs, $\hat{\theta}$ under appropriate regularity conditions, are consistent and asymptotically normally distributed with means θ and variances $\sigma_n^2(\hat{\theta})$. Consequently, the asymptotic two-sided $100(1-\gamma)\%$ confidence intervals with an approximate confidence coefficient $1-\gamma$ for MLEs of a population parameters θ can be constructed such that,

$$P\left(\hat{\theta} - Z_{\frac{\gamma}{2}}\sqrt{var(\hat{\theta})} \le \theta \le \hat{\theta} + Z_{\frac{\gamma}{2}}\sqrt{var(\hat{\theta})}\right) \cong \gamma$$

where $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ the percentile of standard normal distribution. Therefore, the two-sided approximate $100(1-\gamma)\%$ confidence limits for λ, α and β are given, respectively, as follows:

$$\begin{aligned} L_\lambda &= \hat{\lambda} - z\sigma(\hat{\lambda}), & U_\lambda &= \hat{\lambda} + z\sigma(\hat{\lambda}) \\ L_\alpha &= \hat{\alpha} - z\sigma(\hat{\alpha}), & U_\alpha &= \hat{\alpha} + z\sigma(\hat{\alpha}) \\ L_\beta &= \hat{\beta} - z\sigma(\hat{\beta}), & U_\beta &= \hat{\beta} + z\sigma(\hat{\beta}) \end{aligned} \tag{5.1}$$

Therefore, the two sided approximate confidence limits for model parameters under different sample sizes will be constructed using equation (5.1) with confidence levels 95% and 99%.

VI. Simulation Procedure

Monte Carlo simulation study is carried out to illustrate the theoretical results of both estimation and optimal design problems. The performance of the resulting estimators of the parameters has been considered in terms of their absolute relative bias (ARB), mean square error (MSE) and relative error (RE). Furthermore, the asymptotic variance and covariance matrix and optimum test plans are developed. The two-sided confidence intervals of the model parameters are obtained. The Simulation procedures are described through the following steps

Step 1: A random samples of sizes 100 (100) 500 were generated from inverse Weibull distribution. This can be achieved by using the transformation

$$y_i = [-\frac{1}{\lambda}\ln u_i]^{-\frac{1}{\alpha}}, \qquad i = 1,2,\dots,n$$

where, $u_1, u_2, \dots, u_n$ are a random sample from uniform (0,1).

The chosen parameters values are selected as $(\lambda = 2, \alpha = 0.5, \beta = 1.5)$, $(\lambda = 1.5, \alpha = 1, \beta = 1.3)$, $(\lambda = 3, \alpha = 1.5, \beta = 1.1)$ and $(\lambda = 2.5, \alpha = 1.7, \beta = 1.2)$.

Step 2: Under type II censoring, choose a proportion of test units failing at normal condition to be $\pi = 10\%$ and the number of failure units r =0.85 n (the test will terminate after 85% of the test units failed), where the censoring time of a FSS-PALT will be y_r.

Step 3: For each sample the acceleration factor and the parameters of the distribution are estimated in failure step stress PALT under type II censored sample.

Step 4: Repeat the previous steps from 1 to 3 N times representing N different samples, where N=1000.

Step 5: Newton-Raphson method is used for solving the three nonlinear equations (3.3), (3.4) and (3.5) respectively, to obtain the estimators of λ, α and β.

Step 6: The MSE, ARB and RE of $\hat{\lambda}, \hat{\alpha}$ *and* $\hat{\beta}$ over the 1000 samples are obtained.

Step 7: Calculate the Fisher-information matrix then inverted to get the asymptotic variance and covariance matrix of the estimators for different sample sizes.

Step 8: The two sided confidence limit with confidence level $\gamma = 0.95$ and $\gamma = 0.99$ for the acceleration factor and the distribution parameters are constructed.

Step 9: The GAV is calculated from the Fisher-information by solving equation (4.1).

Step 10: Obtained the optimum proportion of units π^* that must fail at normal condition by numerically solving equation (4.4) using the results of step 5. In addition, the value of $n_1{}^*$ (the optimum number of failure units at use condition) is calculated.

VII. SIMULATION RESULTS

Simulation results are presented in Tables 1, 2, 3 and 4.

1. Table 1 gives the MSE, ARB and RE of the estimators. The MSEs, ARBs and REs for the parameters cases $(\lambda = 3, \alpha = 1.5, \beta = 1.1)$ and $(\lambda = 2.5, \alpha = 1.7, \beta = 1.2)$ have a good statistical properties than the parameters cases $(\lambda = 2, \alpha = 0.5, \beta = 1.5)$ and $(\lambda = 1.5, \alpha = 1, \beta = 1.3)$ and for all sample sizes. As the sample size increases the MSEs, ARBs and REs of the estimated parameters decrease. This indicates that the maximum likelihood estimates provide asymptotically normally distributed and consistent estimators for the parameters and acceleration factor.
2. The asymptotic variances and covariance matrix of the estimators are displayed in Table 2. The asymptotic variances of the estimators are decreasing when the sample size is increasing.
3. Table 3 presents the approximated two-sided confidence limits at 95 % and 99% for the model parameters and acceleration factor. The interval of the estimators decreases when the sample size is increasing. Also, the interval of the estimator at $\gamma = 0.95$ is shorter than the interval of estimator at $\gamma = 0.99$.
4. Table 4 gives the optimum switch point π^*, the value of $n_1{}^*$ and GAV. As the sample size increases, the GAVs are decreasing. The optimum value of π^* for the case parameters$(\lambda = 2, \alpha = 0.5, \beta=1.5$, is approximately 90%, that means, the optimum number of units that should fail at used condition to switch to the accelerate condition is $n_1{}^* = 0.90 \times n$. This indicates that all units tend to fail at normal use condition, i.e., testing only at normal condition. In the parameters cases $(\lambda = 1.5, \alpha=1, \beta=1.3, \lambda=3, \alpha=1.5, \beta=1.1$ and $\lambda=2.5, \alpha=1.7, \beta=1.2$ the optimum value of $\pi*$ is approximately 40%, that means, the optimum number of units that should fail at use condition to switch to accelerated condition is $n_1{}^* = 0.40 \times n$.

Table 1: The MSE, ARB and RE of the estimators of the parameters (λ, α, β) for different sized samples

n	Parameter (λ, α, β)	Case 1 (λ = 2 , α = 0.5 , β = 1.5)			Case 2 (λ = 1.5 , α = 1 , β = 1.3)		
		MSE	ARB	RE	MSE	ARB	RE
100	λ	0.016	0.063	0.063	0.013	0.095	0.095
	α	0.004	0.123	0.124	0.007	0.076	0.078
	β	0.338	0.445	0.447	0.064	0.280	0.281
200	λ	0.011	0.053	0.053	0.013	0.077	0.077
	α	0.001	0.066	0.067	0.000	0.010	0.010
	β	0.319	0.435	0.435	0.057	0.266	0.266
300	λ	0.011	0.052	0.052	0.007	0.060	0.060
	α	0.000	0.040	0.040	0.000	0.010	0.010
	β	0.285	0.410	0410	0.048	0.261	0.261
400	λ	0.007	0.042	0.042	0.001	0.022	0.022
	α	0.000	0.040	0.000	0.000	0.006	0.006
	β	0.279	0.410	0.410	0.044	0.244	0.244
500	λ	0.001	0.021	0.021	0.001	0.018	0.018
	α	0.000	0.040	0.000	0.000	0.006	0.006
	β	0.205	0.410	0.410	0.027	0.205	0.206
n	Parameter	Case 3			Case 4		

	(λ, α, β)	(λ = 3 , α = 1.5 , β = 1.1)			(λ = 2.5 , α = 1.7 , β = 1.2)		
		MSE	ARB	RE	MSE	ARB	RE
100	λ	0.008	0.027	0.029	0.042	0.082	0.082
	α	0.005	0.041	0.045	0.034	0.104	0.109
	β	0.033	0.162	0.166	0.114	0.278	0.281
200	λ	0.007	0.027	0.028	0.017	0.052	0.052
	α	0.003	0.041	0.042	0.019	0.079	0.081
	β	0.024	0.139	0.140	0.114	0.270	0.280
300	λ	0.003	0.019	0.019	0.002	0.021	0.021
	α	0.003	0.030	0.030	0.009	0.051	0.052
	β	0.011	0.104	0.105	0.086	0.245	0.244
400	λ	0.002	0.015	0.015	0.000	0.006	0.006
	α	0.003	0.030	0.030	0.001	0.015	0.016
	β	0.001	0.039	0.040	0.015	0.150	0.151
500	λ	0.000	0.004	0.004	0.000	0.004	0.005
	α	0.002	0.030	0.030	0.000	0.002	0.005
	β	0.001	0.030	0.031	0.004	0.082	0.083

Table 2: Asymptotic variance and covariance of estimators

n	Case 1 (λ = 2 , α = 0.5 , β = 1.5)			Case 2 (λ = 1.5 , α = 1 , β = 1.3)		
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
100	7.112	0.654	5.777	2.588	-0.147	1.034
	-------	0.278	0.804	-------	0.970	-0.108
	-------	-------	9.662	-------	-------	1.784
200	6. 335	0.474	3.590	2.405	-0.155	1.031
	-------	0.238	0.479	-------	0.953	-0.057
	-------	-------	6.987	-------	-------	1.683
300	6.330	0.535	4.239	2.401	-0.193	1.077
	-------	0.236	0.596	-------	0.941	-0.119
	-------	-------	6.699	-------	------	1.610
400	6.010	0.504	4.090	2.374	-0.215	0.872
	-------	0.236	0.538	-------	0.878	-0.101
	-------	-------	5.667	-------	-------	1.486
500	5.572	0.449	2.982	2.230	-0.142	0.977
	-------	0.232	0.425	-------	0.843	-0.051
	-------	------	4.288	-------	------	1.287
n	Case 3 (λ = 3 , α = 1.5 , β = 1.1)			Case 4 (λ = 2.5 , α = 1.7 , β = 1.2)		
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
100	16.978	2.540	3.400	8.170	1.031	1.985
	-------	1.672	0.264	-------	2.605	-0.079
	-------	-------	2.567	-------	-------	2.221
200	16.833	2.510	3.278	8.163	1.194	1.859
	-------	1.665	0.213	-------	2.579	-0.070
	-------	-------	2.561	-------	-------	1.892
300	15.254	2.303	2.824	7.723	1.476	1.442
	-------	1.661	0.178	-------	2.453	-0.001
	-------	-------	2.188	-------	-------	1.241
400	14.906	2.379	2.777	7.588	1.275	1.328
	-------	1.642	0.197	-------	2.244	-0.059
	-------	-------	2.055	-------	-------	1.155
500	14.296	2.225	2.692	7.574	1.221	1.437
	-------	1.612	0.174	-------	2.220	-0.091
	-------	-------	2.018	-------	-------	1.070

Table 3: Confidence Bounds of the parameters at confidence level 0.95 and 0.99

n	Parameter	Case 1 (λ = 2 , α = 0.5 , β = 1.5)			Case 2 (λ = 1.5 , α = 1 , β = 1.3)		
		Standard deviation	Lower Bound	Upper Bound	Standard deviation	Lower Bound	Upper Bound
100	λ	0.014	1.847 1.839	1.902 1.911	0.011	1.363 1.356	1.407 1.414
	α	0.009	0.543 0.537	0.580 0.585	0.019	1.148 1.136	1.221 1.232
	β	0.059	0.606 0.569	0.838 0.875	0.021	0.606 0.593	0.689 0.702
200	λ	0.004	1.887 1.884	1.902 1.904	0.005	1.304 1.301	1.323 1.326
	α	0.002	0.529 0.528	0.537 0.539	0.008	1.089 1.084	1.122 1.127
	β	0.015	0.706 0.697	0.764 0.773	0.010	0.640 0.634	0.681 0.687
300	λ	0.003	1.889 1.887	1.902 1.904	0.005	1.307 1.304	1.326 1.329
	α	0.002	0.510 0.509	0.519 0.521	0.007	1.120 1.116	1.146 1.150
	β	0.017	0.733 0.723	0.800 0.811	0.013	0.655 0.647	0.706 0.714
400	λ	0.003	1.911 1.909	1.923 1.924	0.004	1.263 1.260	1.279 1.282
	α	0.002	0.520 0.519	0.526 0.527	0.005	1.158 1.155	1.178 1.181
	β	0.014	0.745 0.736	0.798 0.807	0.007	0.578 0.574	0.605 0.609
500	λ	0.001	1.835 1.834	1.840 1.841	0.003	1.317 1.315	1.329 1.331
	α	0.001	0.522 0.522	0.526 0.527	0.004	1.153 1.150	1.170 1.173
	β	0.006	0.636 0.632	0.659 0.662	0.006	0.623 0.619	0.648 0.652

Continued Table 3

n	Parameter	Case 3 (λ = 3 , α = 1.5 , β = 1.1)			Case 4 (λ = 2.5 , α = 1.7 , β = 1.2)		
		Standard deviation	Lower Bound	Upper Bound	Standard deviation	Lower Bound	Upper Bound
100	λ	0.028	3.026 3.009	3.136 3.154	0.021	2.256 2.243	2.336 2.349
	α	0.028	1.507 1.490	1.616 1.633	0.055	1.769 1.735	1.984 2.017
	β	0.037	0.850 0.827	0.994 1.016	0.055	0.759 0.724	0.975 1.01
200	λ	0.016	3.052 3.042	3.113 3.123	0.008	2.356 2.351	2.387 2.392
	α	0.014	1.512 1.503	1.568 1.577	0.027	1.782 1.765	1.887 1.903
	β	0.019	0.911 0.899	0.984 0.995	0.021	0.821 0.808	0.903 0.916
300	λ	0.007	2.929 2.925	2.958 2.962	0.004	2.341 2.3	2.355 2.322

	α	0.008	1.547 1.542	1.578 1.583	0.012	1.905 1.898	1.95 1.957
	β	0.010	0.875 0.869	0.916 0.922	0.007	0.693 0.688	0.72 0.725
400	λ	0.005	2.945 2.942	2.964 2.967	0.004	2.278 2.275	2.294 2.296
	α	0.006	1.545 1.541	1.566 1.570	0.011	1.923 1.916	1.967 1.974
	β	0.007	0.853 0.849	0.876 0.880	0.007	0.665 0.66	0.695 0.699
500	λ	0.004	2.880 2.877	2.897 2.900	0.004	2.303 2.339	2.319 2.357
	α	0.004	1.543 1.540	1.560 1.562	0.008	1.881 1.876	1.912 1.917
	β	0.005	0.864 0.861	0.881 0.884	0.005	0.724 0.721	0.744 0.747

Table 4: The results of optimal design of failure step-stress PALT

n	Case 1 ($\lambda = 2, \alpha = 0.5, \beta = 1.5$)			Case 2 ($\lambda = 1.5, \alpha = 1, \beta = 1.3$)		
	π^*	n_1^*	GAV	π^*	n_1^*	GAV
100	0.886	89	0.072	0.393	40	0.030
200	0.908	182	0.020	0.340	68	0.012
300	0.904	271	0.015	0.352	106	0.009
400	0.906	363	0.012	0.357	143	0.005
500	0.909	455	0.007	0.358	179	0.005
n	Case 3 ($\lambda = 3, \alpha = 1.5, \beta = 1.1$)			Case 4 ($\lambda = 2.5, \alpha = 1.7, \beta = 1.2$)		
	π^*	n_1^*	GAV	π^*	n_1^*	GAV
100	0.367	37	0.393	0.479	48	0.263
200	0.334	67	0.195	0.403	81	0.118
300	0.325	98	0.105	0.359	108	0.055
400	0.314	126	0.072	0.369	148	0.036
500	0.314	157	0.056	0.353	177	0.032

VIII. CONCLUSION

This article considered the problem of optimally designing simple failure step stress PALT plans under type II censoring. The test unit was assumed to follow inverse Weibull distribution. Optimal test plans are important to improve the accuracy of parameter estimation thereby improving the quality of inference. For this, the optimum of these plans is more useful and efficient to estimate the life distribution at design stress. The performance of FSS-PALT plans is usually evaluated by the MLE model assumptions of the model parameters. The asymptotic variance and covariance of estimators are obtained. Based on asymptotic normality, the two sided confidence limits of the model parameters are constructed. The optimal test plans were obtained by minimizing the GAV of the MLEs of the model parameters. It is noted via the optimal value of π^* , which is the proportion of units that must fail at use stress for switching to accelerated stress.

As shown, from the numerical results, as the sample size increases, the ARBs, MSEs and REs of the model parameters decrease. The interval of estimators decreases when the sample size increases. Also, the optimum GAV decreases when the sample size increases. Finally, it seems that, for most parameter cases, the optimal proportion of the test units that must fail at use stress for switching to accelerated stress is approximately 40%.

FSS-PALT requires constant monitoring of the units under test and may not be convenient. But it is more appropriate than TSS-PALT, where it enables the experimenter to collect sufficient information and to make a good statistical inference about the population parameters. This makes the prediction reliability with highly level of accuracy.

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