Solution of Porous Medium Equation by Homotopy Perturbation Transform Method

Deepti Mishra[1], V.H Pradhan[1], M.N Mehta[1]
[1]Department of Mathematics, S.V National Institute of Technology, Surat-395007

Abstract
In this paper, we apply the Homotopy Perturbation Transform Method (HPTM) using He's polynomials for finding the analytical solution of porous medium equation. The proposed method is an elegant combination of Laplace transform method and the Homotopy perturbation methods. The suggested algorithm is quite efficient and is practically well suited for use in such problems. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. The results reveal that this method is very efficient, simple and can be applied to other nonlinear problems.

Keywords: He's polynomial, Laplace transform method, Homotopy perturbation method, Porous medium equation

1. Introduction

The porous medium equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{m}\frac{\partial u}{\partial x}\right) \tag{1}$$

where m is a rational number, is a prominent example of nonlinear partial differential equation [8]. In the particular case $m=2$ it leads to Boussinesq equation. This equation is one of the simplest examples of nonlinear evolution equation of parabolic type. It appears in the description of different natural phenomena, and its theory and properties depart strongly from the heat equation, $u_t=\Delta u$, its most famous relative. There are a number of physical applications where this simple equation appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. May be the best known of them is the description of the flow of an isentropic gas through a porous medium, modelled independently by Leibenzon and Muskat around 1930. An earlier application is found in the study of groundwater infiltration by Boussisnesq in 1903. Another important application refers to heat radiation in plasmas, developed by Zel'dovich and co-workers around 1950. Indeed, this application was at the base of the rigorous mathematical development of the theory. Other applications have been proposed in mathematical biology, spread of viscous fluids, boundary layer theory, and other fields.

With the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. Several techniques including the Adomain's decomposition method, the Variational iteration method, the weighted finite difference method, the Laplace decomposition method and the Variational iteration decomposition method have their own limitation like the calculation of Adomain's polynomials and the Lagrange's multipliers. The results obtained by these methods are divergent in most cases and which results in causing a lot of chaos. To overcome these difficulties and drawbacks such new techniques are introduced for finding the approximate results. Motivated and inspired by the ongoing research in these areas, we consider a new method, which is called the Homotopy perturbation transform method (HPTM). The suggested HPTM provides the solution in a rapid convergent series which may leads the solution in a closed form. The advantage of this method is its capability of combining of two powerful methods for obtaining exact solution for nonlinear equations. The use of He's polynomials in the nonlinear term was first introduced by Gorbhani . It is worth mentioning that the HPTM is applied without any discretization or restrictive assumptions or transformations and free from round-off errors.

2. Method

To illustrate the basic idea of this method [3,1],we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form

$$\begin{aligned} &Du(x,t)+Ru(x,t)+Nu(x,t)=g(x,t)\\ &u(x,0)=h(x),\quad u_t(x,0)=f(x)\end{aligned} \tag{2}$$

Where D is the second order linear differential operator $D=\frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represent the general non-linear differential operator and $g(x,t)$ is the source term. Taking the Laplace transform (denoted by L) on both side of Eq(2):

$$L[Du(x,t)]+L[Ru(x,t)]+L[Nu(x,t)]=L[g(x,t)] \tag{3}$$

Using the differentiation property of the Laplace transform,[2] we have

$$L[u(x,t)]=\frac{h(x)}{s}+\frac{f(x)}{s^2}-\frac{1}{s^2}L[Ru(x,t)]+\frac{1}{s^2}L[g(x,t)]-\frac{1}{s^2}L[Nu(x,t)] \tag{4}$$

Operating with the Laplace inverse on both side of Eq.(4) gives

$$u(x,t)=G(x,t)-L^{-1}\left[\frac{1}{s^2}L\left[Ru(x,t)+Nu(x,t)\right]\right] \tag{5}$$

Where $G(x,t)$ represent the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method [7,9]

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t) \tag{6}$$

And the nonlinear term can be decomposed as

$$Nu(x,t)=\sum_{n=0}^{\infty}p^n H_n(u) \tag{7}$$

For some He's polynomial H_n that are given by

$$H_n(u_0......u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}(p^i u_i)\right)\right]_{p=0}, \; n=0,1,2,3,.....$$

Substituting Eqs. (7) and (6) in Eq. (5) we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=G(x,t)-p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)+\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right) \tag{8}$$

This is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomial. Comparing the coefficient of like powers of p, the following approximations are obtained [4]

$$\begin{aligned}
p^0 &: u_0(x,t)=-\frac{1}{s^2}L\left[Ru_0(x,t)+H_0(u)\right]\\
p^1 &: u_1(x,t)=-\frac{1}{s^2}L\left[Ru_1(x,t)+H_0(u)\right]\\
p^2 &: u_2(x,t)=-\frac{1}{s^2}L\left[Ru_1(x,t)+H_1(u)\right]\\
p^3 &: u_3(x,t)=-\frac{1}{s^2}L\left[Ru_2(x,t)+H_2(u)\right]
\end{aligned} \tag{9}$$

The best approximations for the solutions are

$$u=\lim_{p\to 1}u_n=u_0+u_1+u_{2+..........} \tag{10}$$

4. Applications

Example 1. Let us take $m=-1$ in equation (1), we get

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left((u)^{-1}\frac{\partial u}{\partial x}\right) \tag{11}$$

With initial condition as $u(x,0)=\frac{1}{x}$

Exact solution [4] of this equation is $u(x,t)=\left(c_1x-c_1^2t+c_2\right)^{-1}$ with the values of arbitrary constants taken as $c_1=1$ and $c_2=0$ solution becomes

$$u(x,t)=\frac{1}{x-t} \tag{12}$$

Using HPTM we can find solution by applying Laplace transform on both side of equation (11) subject to the initial condition

$$L\left[\frac{\partial u}{\partial t}\right]=L\left[\left(u^{-1}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)\right]-L\left[\left(u^{-2}\right)\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{13}$$

This can be written as

$$[su(x,s)-u(x,0)]=L\left[\left(u^{-1}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)\right]-L\left[\left(u^{-2}\right)\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{14}$$

On applying the above specified initial condition we get

$$su(x,s)-\left(\frac{1}{x}\right)=L\left[\left(u^{-1}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)\right]-L\left[\left(u^{-2}\right)\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{15}$$

$$u(x,s)=\frac{1}{s}\left(\frac{1}{x}\right)+\frac{1}{s}L\left[\left(u^{-1}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)\right]-\frac{1}{s}L\left[\left(u^{-2}\right)\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{16}$$

Taking Inverse Laplace Transform on both sides we get

$$L^{-1}\left[u(x,s)\right]=L^{-1}\left[\frac{1}{s}L\left[\left(u^{-1}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)\right]\right]-L^{-1}\left[\frac{1}{s}L\left[\left(u^{-2}\right)\left(\frac{\partial u}{\partial x}\right)^2\right]\right] \tag{17}$$

$$u(x,t)=L^{-1}\left[\frac{1}{s}L\left[\left(u^{-1}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)\right]\right]-L^{-1}\left[\frac{1}{s}L\left[\left(u^{-2}\right)\left(\frac{\partial u}{\partial x}\right)^2\right]\right] \tag{18}$$

Now we apply the homotopy perturbation method in the form

$$u(x,t)=\sum_{n=0}^{\infty}p^n\left(u_n(x,t)\right) \tag{19}$$

Using Binomial expansion and He's Approximation, equation (18) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty}p^n\left(u_n(x,t)\right)=&L^{-1}\left[\frac{1}{s}L\left[\left(\left(\sum_{n=0}^{\infty}p^n\left(u_n(x,t)\right)\right)^{-1}\right)\left(\left(\sum_{n=0}^{\infty}p^n\left(u_n(x,t)\right)\right)_{xx}\right)\right]\right]\\&-L^{-1}\left[\frac{1}{s}L\left[\left(\left(\sum_{n=0}^{\infty}p^n\left(u_n(x,t)\right)\right)^{-2}\right)\left(\left(\sum_{n=0}^{\infty}p^n\left(u_n(x,t)\right)\right)_x\right)^2\right]\right]\end{aligned} \tag{20}$$

This can be written in expanded form as

$$u_0 + pu_1 + p^2u_2 + ... = \left(\frac{1}{x}\right) + pL^{-1}\left[\frac{1}{s}L\left[\begin{matrix}\left((u_0 + pu_1 + p^2u_2 + ..)\right)^{-1} \\ \left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + ...\right)\end{matrix}\right]\right]$$

$$-pL^{-1}\left[\frac{1}{s}L\left[\left((u_0 + pu_1 + p^2u_2 + ..)\right)^{-2}\left(\left(\frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x} + p^2\frac{\partial u_2}{\partial x} + ...\right)^2\right)\right]\right] \quad (21)$$

Comparing the coefficient of various power of p, we get

$$p^0 : u_0(x,t) = \frac{1}{x}$$

$$p^1 : u_0(x,t) = L^{-1}\left[\frac{1}{s}L\left[(u_0)^{-1}\left(\frac{\partial^2 u_0}{\partial x^2}\right)\right]\right] - L^{-1}\left[\frac{1}{s}L\left[\left((u_0)^{-2}\left(\frac{\partial u_0}{\partial x}\right)^2\right)\right]\right] \quad (22)$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s}L\left[(u_0)^{-1}\left(\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2}\left(\frac{u_1}{u_0}\right)\right)\right]\right] - L^{-1}\left[\frac{1}{s}L\left[(u_0)^{-2}\left(2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right) - 2\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{u_1}{u_0}\right)\right)\right]\right]$$

Proceeding in similar manner we can obtain further values, substituting above values in equation (10) we get solution in the form of a series

$$u(x,t) = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4} + ... \quad (23)$$

Which is the exact solution obtained in (12) in the closed form.

Example 2. When $m = 1$, Eq. (1) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) \quad (24)$$

With initial condition $u(x,0) = x$

Apply Laplace transform on both the sides of Eq. (24) subject to the initial condition

$$L\left[\frac{\partial u}{\partial t}\right] = L\left[\left(\frac{\partial u}{\partial x}\right)^2\right] + L\left[u\frac{\partial^2 u}{\partial x^2}\right] \quad (25)$$

This can be written on applying the above specified initial condition as

$$u(x,s) = \frac{1}{s}(x) + \frac{1}{s}L\left[\left(\frac{\partial u}{\partial x}\right)^2\right] + \frac{1}{s}L\left[\left(u\frac{\partial^2 u}{\partial x^2}\right)\right] \quad (26)$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}[u(x,s)] = xL^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s}L\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(u\frac{\partial^2 u}{\partial x^2}\right)\right]\right] \quad (27)$$

$$u(x,t) = x + L^{-1}\left[\frac{1}{s}L\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(u\frac{\partial^2 u}{\partial x^2}\right)\right]\right] \quad (28)$$

Now on applying the homotopy perturbation method in the form

$$u(x,t) = \sum_{n=0}^{\infty} p^n\left(u_n(x,t)\right) \quad (29)$$

Equation (28) can be reduces to

$$\sum_{n=0}^{\infty} p^n\left(u_n\left(x,t\right)\right) = x + L^{-1}\left[\frac{1}{s}L\left[\begin{array}{l}\left(\left(\sum_{n=0}^{\infty} p^n\left(u_n\left(x,t\right)\right)\right)_x\right)^2 + \\ \left(\left(\sum_{n=0}^{\infty} p^n\left(u_n\left(x,t\right)\right)\right)\left(\sum_{n=0}^{\infty} p^n\left(u_n\left(x,t\right)\right)\right)_{xx}\right)\end{array}\right]\right] \tag{30}$$

On expansion of equation (30) and comparing the coefficient of various powers of p, we get

$$p^0 : u_0(x,t) = x$$

$$p^1 : u_0(x,t) = L^{-1}\left[\frac{1}{s}L\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right]\right] + L^{-1}\left[\frac{1}{s}L\left[\left(u_0\frac{\partial^2 u_0}{\partial x^2}\right)\right]\right]$$

$$p^2 : u_2\left(x,t\right) = L^{-1}\left[\frac{1}{s}L\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)\right]\right] + L^{-1}\left[\frac{1}{s}L\left[\left(u_1\frac{\partial^2 u_0}{\partial x^2} + u_0\frac{\partial^2 u_1}{\partial x^2}\right)\right]\right] \tag{31}$$

In this case the values obtained as $u_0 = x$, $u_1 = t$ and $u_2 = 0$ which follows $u_n\left(x,t\right) = 0$ for $n \geq 2$. Putting these values in (10) we get the solution as

$$u\left(x,t\right) = x + t \tag{32}$$

This is same as the exact solution given in [6]

Example 3. Let $m = -4/3$, Eq. (1) attains

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right) \tag{33}$$

The exact solution to Eq.(33) is given by [5]

$u\left(x,t\right) = \left(2c_1x - 3c_1^2t + c_2\right)^{-3/4}$, where $c_1 = 1$ and $c_2 = 0$ is taken for simplicity, it reduces to

$$u\left(x,t\right) = \left(2x - 3t\right)^{-3/4} \tag{34}$$

Applying the same procedure again on Eq. (33) subject to the initial condition $u\left(x,0\right) = \left(2x\right)^{-3/4}$ (35)

We get, $p^0 : u_0(x,t) = \left(2x\right)^{-3/4}$

$$p^1 : u_0(x,t) = L^{-1}\left[\frac{1}{s}L\left[\left(u_0\right)^{-4/3}\left(\frac{\partial^2 u_0}{\partial x^2}\right)\right]\right] - L^{-1}\left[\frac{1}{s}L\left[\left(\frac{4}{3}\right)\left(\left(u_0\right)^{-7/3}\left(\frac{\partial u_0}{\partial x}\right)^2\right)\right]\right]$$

$$p^2 : u_2\left(x,t\right) = L^{-1}\left[\frac{1}{s}L\left[\left(u_0\right)^{-4/3}\left(\frac{\partial^2 u_1}{\partial x^2} - \left(\frac{4}{3}\right)\frac{\partial^2 u_0}{\partial x^2}\left(\frac{u_1}{u_0}\right)\right)\right]\right]$$
$$-L^{-1}\left[\frac{1}{s}L\left[\left(\frac{4}{3}\right)\left(u_0\right)^{-7/3}\left(2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right) - \frac{7}{3}\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{u_1}{u_0}\right)\right)\right]\right] \tag{36}$$

On solving above we get values as $u_0 = \left(2x\right)^{-3/4}$, $u_1 = 9\times 2^{-15/4}\times x^{-7/4}\times t$, $u_2 = 189\times 2^{-31/4}\times x^{-11/4}\times t^2$ and so on. Substituting these terms in Eq. (11), one obtains

$$u\left(x,t\right) = u_0\left(x,t\right) + u_1\left(x,t\right) + u_2\left(x,t\right) + ...$$
$$= \left(2x\right)^{-3/4} + 9\times 2^{-15/4}x^{-7/4}t + 189\times 2^{-31/4}x^{-11/4}t^2 + ... \tag{37}$$

This gives the exact solution obtained in Eq. (34) in the closed form.

Conclusion

The main concern of this article is to construct an analytical solution for Porous medium equation with different powers of m. We have achieved this goal by applying homotopy perturbation transform method (HPTM). The main advantage of this algorithm is the fact that it provides its user with an analytical approximation, in many cases an exact solution in a rapidly convergent sequence with elegantly computed terms. Its small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence show that the method is reliable and introduces a significant improvement in solving partial differential equations over existing methods. Finally, we conclude that HPTM can be considered as a nice refinement in existing numerical techniques and might find wide application in different field of sciences.

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