Solution of Burger's equation and coupled Burger's equations by Homotopy perturbation method

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Abstract
In the present paper, the exact solution of Burger's equation and coupled Burger's equation is obtained using Homotopy Perturbation Method(HPM). The solutions obtained are found exactly the same as with the Adomian decomposition method. He's Homotopy perturbation method is introduced to overcome the difficulty arising in calculating Adomian polynomials. HPM is found more effective than the later.

Keywords: Burger's equation, Homotopy perturbation method, nonlinear partial differential equations.

1. Introduction

Burger's equation arises in a number of physically important phenomena such as model of traffic, turbulence, shock waves and fluid flow[3]. Many authors Bateman H[2], Burger J.M[3], Cole, J.D[5], Mittal R.C and Singhal P[13], Dogan A[6], Aksan, E.N., A. Özdes and T. Özis [6], Caldwell, J., P. Wanless and A.E. Cook [7] have discussed the numerical solution of Burger's equation using Finite Difference Methods and Finite Element Methods. have obtained the exact solution of Burger's equation by using Adomian Decomposotion method. The main difficulty found in calculating Adomian polynomials is overcome by using He's Homotopy Perturbation Method and the solution obtained is exactly the same.

The Homotopy perturbation method was first proposed by He [10,11] and was successfully applied to ordinary differential equations, to nonlinear polycrystalline solids and other fields. The Homotopy perturbation method is a combination of traditional Perturbation method and Homotopy method. This method has much advantages such as apply directly to the non linear partial differential equation without linearizing the problem and easy to calculate the solution. In this paper the Homotopy perturbation method proposed by He is extended to solve the one-dimensional Burger's equation and coupled Burger's equations.

2. Homotopy perturbation method

To explain this method let us consider the following function:

$$A(u) - f(r) = 0 \quad r \in \Omega \qquad (1.1)$$

With boundary conditions of

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0 \qquad (1.2)$$

where A , B , $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω respectively. Generally speaking, the operator A can be divided into a linear part L(u) and non linear part N(u). So equation (1.1) may written as

$$L(u) + N(u) - f(r) = 0 \qquad (1.3)$$

By Homotopy technique, we construct a Homotopy
$v(r, p) : \Omega \times [0,1] \to R$ which satisfies[11]:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \qquad (1.4)$$

$p \in [0,1], \quad r \in \Omega$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{1.5}$$

Where $p \in [0,1]$ is an embedding parameter, while u_0 is initial approximation of equation (1.1) which satisfies the boundary conditions from equations (1.4) and (1.5) we will have,

$$H(v,0) = L(v) - L(u_0) = 0 \tag{1.6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{1.7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to u(r). In topology this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called Homotopy. According to Homotopy perturbation method, we can first use the embedding parameter p as a "small parameter" and assume that the solution of equation (1.4) and (1.5) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{1.8}$$

Setting p=1 yields in the approximate solution of equation (1.8) to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{1.9}$$

Equation (1.9) is the solution of equation (1.1) obtained by Homotopy perturbation method.

3. Applications

Problem – 1

Burger's equation

Consider the one dimensional Burger's equation has the form [9,12]

$$u_t + uu_x - \varepsilon u_{xx} = 0 \tag{1.10}$$

With initial condition

$$u(x,0) = \frac{\alpha + \beta + (\beta - \alpha)\exp(\gamma)}{1 + \exp(\gamma)}, \qquad t \ge 0 \tag{1.11}$$

Where $\gamma = \left(\alpha/\varepsilon\right)(x - \lambda)$ and the parameters α, β, γ and v are arbitrary constants.

To solve equation (1.10) using Homotopy perturbation method construct a Homotopy $v(r,p) : \Omega \times [0,1] \to R$ that satisfies[11]

$$H(v,p) = (1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial v_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \varepsilon\frac{\partial^2 v}{\partial x^2}\right) = 0 \tag{1.12}$$

Suppose the solution of the Homotopy given by equation (1.12) can be written as

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + p^4 v_4 + ... \tag{1.13}$$

Using this solution in equation (1.12) we get,

$$(1-p)\left(\frac{\partial v_0}{\partial t}+p\frac{\partial v_1}{\partial t}+p^2\frac{\partial v_2}{\partial t}+p^3\frac{\partial v_3}{\partial t}+p^4\frac{\partial v_4}{\partial t}+\ldots-\frac{\partial v_0}{\partial t}\right)$$
$$+p\begin{pmatrix}\frac{\partial v_0}{\partial t}+p\frac{\partial v_1}{\partial t}+p^2\frac{\partial v_2}{\partial t}+p^3\frac{\partial v_3}{\partial t}+\ldots\\ +\left(v_0+pv_1+p^2v_2+p^3v_3+\ldots\right)\left(\frac{\partial v_0}{\partial x}+p\frac{\partial v_1}{\partial x}+p^2\frac{\partial v_2}{\partial x}+p^3\frac{\partial v_3}{\partial x}+\ldots\right)\\ -\varepsilon\left(\frac{\partial^2 v_0}{\partial x^2}+p\frac{\partial^2 v_1}{\partial x^2}+p^2\frac{\partial^2 v_2}{\partial x^2}+p^3\frac{\partial^2 v_3}{\partial x^2}+\ldots\right)\end{pmatrix}=0 \quad (1.14)$$

Comparing coefficients of terms with identical powers of p leads to,

$$p^0:\frac{\partial v_0}{\partial t}-\frac{\partial v_0}{\partial t}=0 \quad (1.15)$$

$$p^1:\frac{\partial v_1}{\partial t}-\frac{\partial v_0}{\partial t}+\frac{\partial v_0}{\partial t}+\frac{\partial v_0}{\partial t}+v_0\frac{\partial v_0}{\partial x}-\varepsilon\frac{\partial^2 v_0}{\partial x^2}=0 \quad (1.16)$$

$$p^2:\frac{\partial v_2}{\partial t}-\frac{\partial v_1}{\partial t}+\frac{\partial v_1}{\partial t}+v_0\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_0}{\partial x}-\varepsilon\frac{\partial^2 v_1}{\partial x^2}=0 \quad (1.17)$$

$$p^3:\frac{\partial v_3}{\partial t}-\frac{\partial v_2}{\partial t}+\frac{\partial v_2}{\partial t}+v_0\frac{\partial v_2}{\partial x}+v_1\frac{\partial v_1}{\partial x}+v_2\frac{\partial v_0}{\partial x}-\varepsilon\frac{\partial^2 v_2}{\partial x^2}=0 \quad (1.18)$$

$$p^4:\frac{\partial v_4}{\partial t}-\frac{\partial v_3}{\partial t}+\frac{\partial v_3}{\partial t}+v_0\frac{\partial v_3}{\partial x}+v_1\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_1}{\partial x}+v_3\frac{\partial v_0}{\partial x}-\varepsilon\frac{\partial^2 v_3}{\partial x^2}=0 \quad (1.19)$$

Solving all the above nonlinear partial differential equations we get,

$$v_0=\frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)} \quad (1.20)$$

$$v_1=\frac{2\alpha^2\beta\exp(\gamma)}{\varepsilon[1+\exp(\gamma)]^2}t \quad (1.21)$$

$$v_2=\frac{\alpha^3\beta^2\exp(\gamma)\left(-1+\exp(\gamma)\right)}{\varepsilon^2\left[1+\exp(\gamma)\right]^3}t^2 \quad (1.22)$$

$$v_3=\frac{\alpha^4\beta^3\exp(\gamma)\left(1-4\exp(\gamma)+\exp(2\gamma)\right)}{3\varepsilon^3[1+\exp(\gamma)]^4}t^3 \quad (1.23)$$

$$v_4=\frac{\alpha^5\beta^4\exp(\gamma)\left[-1+11\exp(\gamma)-11\exp(2\gamma)+\exp(3\gamma)\right]}{12\varepsilon^4\left[1+\exp(\gamma)\right]^5}t^4 \tag{1.24}$$

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And so on, in same manner further values were obtained using Mathematica.

The solution of equation (1.10), Burger equation is given by,

$$u=v_0+v_1+v_2+v_3+v_4+........ \tag{1.25}$$

$$\begin{aligned}\therefore u=&\frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)}+\frac{2\alpha^2\beta\exp(\gamma)}{\varepsilon\left[1+\exp(\gamma)\right]^2}t\\&+\frac{\alpha^3\beta^2\exp(\gamma)\left(-1+\exp(\gamma)\right)}{\varepsilon^2\left[1+\exp(\gamma)\right]^3}t^2+\frac{\alpha^4\beta^3\exp(\gamma)\left(1-4\exp(\gamma)+\exp(2\gamma)\right)}{3\varepsilon^3\left[1+\exp(\gamma)\right]^4}t^3\\&+\frac{\alpha^5\beta^4\exp(\gamma)\left[-1+11\exp(\gamma)-11\exp(2\gamma)+\exp(3\gamma)\right]}{12\varepsilon^4\left[1+\exp(\gamma)\right]^5}t^4+...\end{aligned} \tag{1.26}$$

The solution of $u(x,t)$ in close form is,

$u(x,t)=\dfrac{\alpha+\beta+(\beta-\alpha)\exp((\alpha/\varepsilon)(x-\beta t-\lambda))}{1+\exp((\alpha/\varepsilon)(x-\beta t-\lambda))}$ which is exactly the same as solution obtained by Adomain decomposition method[7].

The numerical values and behavior of the solutions obtained by Homotopy perturbation method is shown for different values of time in table 1 and figure 1 respectively.

	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5
x=-10	0.98662020746	0.9869323169	0.98723726391	0.98753520741	0.98782630303
x=-9	0.98020254181	0.98066067514	0.98110846349	0.98154612872	0.98197388844
x=-8	0.97082080578	0.97148810816	0.97214070148	0.97277888498	0.9734029529
x=-7	0.95723687696	0.95819798495	0.95913865749	0.96005927714	0.96096022161
x=-6	0.93783912325	0.93920119484	0.94053582601	0.94184346485	0.9431245567
x=-5	0.91068025144	0.91256690989	0.91441851629	0.91623551999	0.91801837324
x=-4	0.87368671730	0.87621867429	0.87870901571	0.88115805958	0.88356613683
x=-3	0.82514137037	0.82839855512	0.83161140956	0.83477992618	0.8379041228
x=-2	0.7644618661	0.76843066294	0.77235955533	0.77624802773	0.78009559821
x=-1	0.69306405178	0.69759136703	0.70209225132	0.70656565079	0.71101054014
x=0	0.61479324996	0.61958432852	0.62436978085	0.62914824141	0.63391835268
x=1	0.5353713615	0.54005491161	0.54475533660	0.54947136712	0.5542017164

x=2	0.460858490319	0.46509516107	0.469366262361	0.473670970737	0.478008430233
x=3	0.39592003356	0.39949227379	0.403107747717	0.406766184875	0.41046728330
x=4	0.34286375002	0.34570193627	0.3485837580	0.35150938028	0.35447894736
x=5	0.30175518481	0.30390582500	0.30609501329	0.30832314789	0.310590618533
x=6	0.271193937309	0.272765965822	0.27436916411	0.27600399292	0.277670912793
x=7	0.24916518145	0.250284354986	0.251427259909	0.252594319391	0.253785960476
x=8	0.233637944017	0.234419865797	0.23521912823	0.236036076702	0.236871061634
x=9	0.2228651493	0.223404302594	0.223955777031	0.224519835556	0.225096745801
x=10	0.21547276358	0.215841159780	0.216218145926	0.216603913583	0.2169986582

Table 1: The values of u(x,t) evaluates by Homotopy perturbation method for different time t with fix value $\varepsilon = 1, \beta = 0.6, \alpha = 0.4$ **[12]**

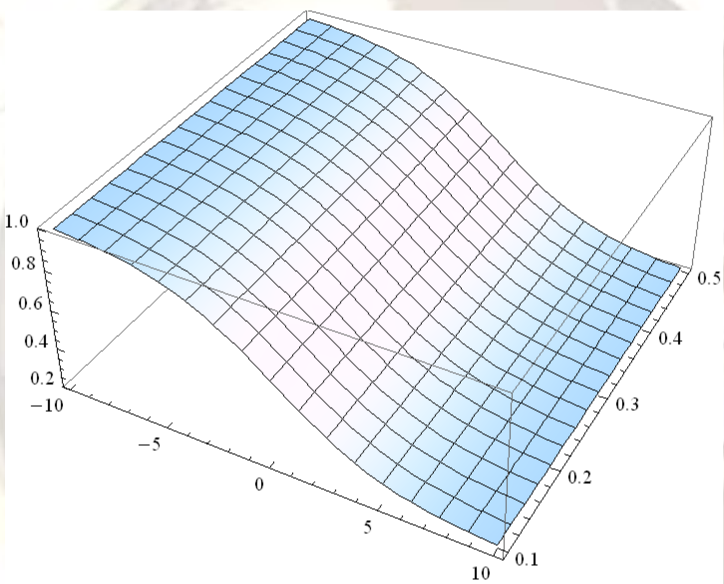

Fig 1: Graphical representation of u(x,t) for different values of time t

Problem – 2
Coupled Burger's equations

To solve the homogeneous form of coupled Burger's equations by Homotopy perturbation method consider the system of equations[14]

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0 \tag{1.26}$$

$$v_t - v_{xx} - 2vv_x + (vu)_x = 0 \tag{1.27}$$

With initial conditions[12]

$$u(x,0) = Sin\ x \qquad\qquad v(x,0) = Sin\ x \tag{1.28}$$

To solve this system of equations by Homotopy perturbation method define Homotopy $h_1(r,p):\Omega\times[0,1]\to R$ and $h_2(r,p):\Omega\times[0,1]\to R$ for equation (1.26) and (1.27) respectively that satisfy

$$H(h_1,p)=(1-p)\left[\frac{\partial h_1}{\partial t}-\frac{\partial u_0}{\partial t}\right]+p\left[\frac{\partial h_1}{\partial t}-\frac{\partial^2 h_1}{\partial x^2}-2h_1\frac{\partial h_1}{\partial x}+h_1\frac{\partial h_2}{\partial x}+h_2\frac{\partial h_1}{\partial x}\right]=0 \tag{1.29}$$

And

$$H(h_2,p)=(1-p)\left[\frac{\partial h_2}{\partial t}-\frac{\partial v_0}{\partial t}\right]+p\left[\frac{\partial h_2}{\partial t}-\frac{\partial^2 h_2}{\partial x^2}-2h_2\frac{\partial h_2}{\partial x}+h_2\frac{\partial h_1}{\partial x}+h_1\frac{\partial h_2}{\partial x}\right]=0 \tag{1.30}$$

Suppose the solutions of the Homotopy (1.29) and (1.30) can be written as

$$h_1=u_0+pu_1+p^2u_2+p^3u_3+p^4u_4+.............. \tag{1.31}$$

and

$$h_2=v_0+pv_1+p^2v_2+p^3v_3+p^4v_4+.............. \tag{1.32}$$

Using these solutions in equations (1.29) and (1.30) and then comparing powers of p on both sides we get,

$$\begin{aligned}
&p^0:\frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0\\
&p^0:\frac{\partial v_0}{\partial t}-\frac{\partial v_0}{\partial t}=0
\end{aligned} \tag{1.33}$$

$$\begin{aligned}
&p^1:\frac{\partial u_1}{\partial t}-\frac{\partial u_0}{\partial t}+\frac{\partial u_0}{\partial t}+\frac{\partial u_0}{\partial t}-\frac{\partial^2 u_0}{\partial x^2}-2u_0\frac{\partial u_0}{\partial x}+u_0\frac{\partial v_0}{\partial x}+v_0\frac{\partial u_0}{\partial x}=0\\
&p^1:\frac{\partial v_1}{\partial t}-\frac{\partial v_0}{\partial t}+\frac{\partial v_0}{\partial t}+\frac{\partial v_0}{\partial t}-\frac{\partial^2 v_0}{\partial x^2}-2v_0\frac{\partial v_0}{\partial x}+v_0\frac{\partial u_0}{\partial x}+u_0\frac{\partial v_0}{\partial x}=0
\end{aligned} \tag{1.34}$$

$$\begin{aligned}
&p^2:\frac{\partial u_2}{\partial t}-\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial t}-\frac{\partial^2 u_1}{\partial x^2}-2u_1\frac{\partial u_0}{\partial x}-2u_0\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_0}{\partial x}\\
&\qquad\qquad+v_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial v_0}{\partial x}+u_0\frac{\partial v_1}{\partial x}=0\\
&p^2:\frac{\partial v_2}{\partial t}-\frac{\partial v_1}{\partial t}+\frac{\partial v_1}{\partial t}-\frac{\partial^2 v_1}{\partial x^2}-2v_1\frac{\partial v_0}{\partial x}-2v_0\frac{\partial v_1}{\partial x}+u_1\frac{\partial v_0}{\partial x}\\
&\qquad\qquad+u_0\frac{\partial v_1}{\partial x}+v_1\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_1}{\partial x}=0
\end{aligned} \tag{1.35}$$

$$\begin{aligned}
&p^3:\frac{\partial u_3}{\partial t}-\frac{\partial u_2}{\partial t}+\frac{\partial u_2}{\partial t}-\frac{\partial^2 u_2}{\partial x^2}-2u_2\frac{\partial u_0}{\partial x}-2u_1\frac{\partial u_1}{\partial x}-2u_0\frac{\partial u_2}{\partial x}\\
&\qquad+v_2\frac{\partial u_0}{\partial x}+v_1\frac{\partial u_1}{\partial x}+v_0\frac{\partial u_2}{\partial x}+u_2\frac{\partial v_0}{\partial x}+u_1\frac{\partial v_1}{\partial x}+u_0\frac{\partial v_2}{\partial x}=0\\
&p^3:\frac{\partial v_3}{\partial t}-\frac{\partial v_2}{\partial t}+\frac{\partial v_2}{\partial t}-\frac{\partial^2 v_2}{\partial x^2}-2v_2\frac{\partial v_0}{\partial x}-2v_1\frac{\partial v_1}{\partial x}-2v_0\frac{\partial v_2}{\partial x}\\
&\qquad+u_2\frac{\partial v_0}{\partial x}+u_1\frac{\partial v_1}{\partial x}+u_0\frac{\partial v_2}{\partial x}+v_2\frac{\partial u_0}{\partial x}+v_1\frac{\partial u_1}{\partial x}+v_0\frac{\partial u_2}{\partial x}=0
\end{aligned} \tag{1.36}$$

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Solving all the above equations we get,

$$u_0 = Sin\ x \qquad u_1 = -tSin\ x$$
$$v_0 = Sin\ x \qquad v_1 = -tSin\ x \tag{1.37}$$

$$u_2 = \frac{t^2}{2!} Sin\ x$$
$$v_2 = \frac{t^2}{2!} Sin\ x \tag{1.38}$$

$$u_3 = -\frac{t^3}{3!} Sin\ x$$
$$v_3 = -\frac{t^3}{3!} Sin\ x \tag{1.39}$$

(1.40)

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And so on further values were obtained.
The solution of coupled Burger's equations can be written as,

$$u = u_0 + u_1 + u_2 + u_3 + ...$$
$$v = v_0 + v_1 + v_2 + v_3 + ... \tag{1.41}$$

$$\therefore u = Sin\, x - tSin\ x + \frac{t^2}{2!} Sin\ x - \frac{t^3}{3!} Sin\ x + ...$$
$$\therefore v = Sin\, x - tSin\ x + \frac{t^2}{2!} Sin\ x - \frac{t^3}{3!} Sin\ x + ... \tag{1.42}$$

$$\therefore u = \exp(-t) Sin\, x$$
$$\therefore v = \exp(-t) Sin\, x \tag{1.43}$$

Which are the exact solutions.

4. Conclusion

A Homotopy perturbation method is successfully applied to solve non linear Burger's equation and coupled Burger's equations. The solution obtained by Homotopy perturbation method is an infinite series for appropriate initial condition that can be expressed in a closed form, the exact solution. The solution obtained by Homotopy perturbation method is found as a powerful mathematical tool to solve non linear partial differential equations.

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