A New Method for Finding an Optimal Solution to Solid Assignment Problems

D.Anuradha and P.Pandian
Department of Mathematics, School of Advanced Sciences
VIT University, Vellore-14, Tamilnadu, India.

Abstract
A new method namely, reduction method is proposed for finding an optimal solution of solid assignment problems. The proposed method does not based on the Hungarian method. With help of a numerical example, the reduction method is illustrated. The solution obtained by the proposed method will help the decision makers to take an appropriate action when they are handling various types of three dimensional assignment problems.

Mathematics Subject Classifications: 90C08, 90C90

Keywords: Solid assignment problem, Optimal solution, The reduction method.

1 Introduction
Assignment problem (AP) is a special type of a transportation problem. An AP can be viewed as a transportation problem in which all supplies and demands equal to one. A solid assignment problem (SAP) is an extension of the assignment problem. Solid assignment problems have wide applications in, multi-passive-sensor, capital investment, dynamic facility location, satellite launching and so on. The SAP was proposed by Pierskalla [6]. Frieze and Yadegar [4] developed an algorithm for solving three-dimensional APs with application to scheduling in a teaching practice. Balas [1] provided an algorithm for the three-index AP. Crama and Spieksma [2] gave approximation algorithms for three dimensional AP with triangle inequalities. Poore [7] showed the application of multidimensional AP. Magos [5] introduced a tabu search for the planar three-index AP. Poore and Robertson [8] discussed a new Lagrangean relaxation based algorithm for a class of multidimensional APs. Storms and Spieksma [9] obtained a solution procedure for geometric three-dimensional APs. Kuroki and Matsui [10] introduced an approximation algorithm for multidimensional assignment problem for minimizing the sum of squared errors. Federico [3] discussed an application of multidimensional AP.

In this paper, we propose a new method namely, reduction method for finding an optimal solution to a solid assignment problem. For quick understanding, the solution procedure by the reduction method is illustrated with a real life example. The proposed method enables the decision makers to evaluate the economical activities and make self satisfied managerial decisions when they are handling a variety of solid dimensional assignment problems.

2. Preliminaries

Consider the following solid assignment problem (SAP):

(P) Minimize $z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk}$

subject to

$$\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} = 1,\quad i = 1, 2, \ldots, m \qquad (1)$$

$$\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} = 1,\quad j = 1, 2, \ldots, n \qquad (2)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} = 1,\quad k = 1, 2, \ldots, l \qquad (3)$$

$x_{ijk} = 0$ (or) 1, for all i, j and k (4)

where c_{ijk} is the cost of assigning job j to be performed by worker i in machine k. $x_{ijk} = 1$, if job j is assigned to worker i in machine k, and $x_{ijk} = 0$, otherwise.

If $m = n = l$, then the SAP is called a balanced SAP. Any set of non-negative allocations to a SAP which satisfies the equations (1), (2), (3) and (4) is called a feasible solution of the SAP. A feasible solution of SAP which minimizes the total assignment cost, that is, $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk}$ is called an optimal solution to the SAP.

3. The Reduction Method

Now, we propose a new method namely, reduction method for finding an optimal solution to SAPs. First, we prove the following theorem which is going to be used in the proposed method.

Theorem 1. Any optimal solution to the problem (Q) where

(Q) Minimize $z^* = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(c_{ijk} - u_i - v_j - w_k)x_{ij}$

subject to (1) to (4) are satisfied ,

where u_i, v_j and w_k are some real values, is an optimal solution to the problem (P) where

(P) Minimize $z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk}$

subject to (1) to (4) are satisfied .

Proof: Now,

$$z^* = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} u_i x_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} v_j x_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} v_j x_{ijk} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} w_k x_{ijk}$$

$$= z - \sum_{i=1}^{m} u_i - \sum_{j=1}^{n} v_j - \sum_{k=1}^{l} w_k \quad \text{(from (1), (2) and (3))}$$

Since $\sum_{i=1}^{m} u_i$, $\sum_{j=1}^{n} v_j$ and $\sum_{k=1}^{l} w_k$ are independent of x_{ijk}, for all i, j and k, we can conclude that any optimal solution to the problem (Q) is also an optimal solution to the problem (P).
Hence the theorem.

Now, we introduce the reduction method for finding an optimal solution to SAP.

The reduction method proceeds as follows:

Step 1: Convert the given SAP into a balanced one, if it is not.

Step 2: Construct Men–Job (M–J) table in which rows are Men's and columns are Jobs.

Step 3: Subtract each entries of Men of the table by its minimum and then, subtract each entries of a Job of the reduced table by its minimum.

Step 4: Check if there is a possibility to assign each Men with a Job which correspond to a cell of zero cost. If so, go to Step 6. If not, go to Step 5.

Step 5: Cover all men(s), job(s) and factory(s) having zeros of reduced assignment table with the minimum number of horizontal and vertical lines. Then, select a smallest uncovered element and subtract this minimum element from all uncovered elements and add the same to all elements at the intersection cells. Then, go to Step 4.

Step 6: Construct the Job–Factory (J–F) table using the reduced table obtained from Step 4. and then, apply Step 4. to Step 5. to the J–F table. Then, go to Step 7.

Step 7: Construct the Factory–Men (F–M) table using the reduced table obtained from Step 6. and then, apply Step 4. to Step 5. to the F–M table. Then, go to Step 8.

Step 8: Check if there is a possible to assign each job with the corresponding men's using the cells having zero cost and each men with the corresponding jobs using the cells having zero cost, each job with the corresponding factories using the cells having zero cost and each factory with the corresponding jobs using the cells having zero and each men with the corresponding factories using the cells having zero cost and each factory with the corresponding men's using the cells having zero of the reduced solid assignment table (Such table is called allocation table). If not, convert into it by applying Step 5.

Step 9: Select a men/a job/a factory having minimum number of zeros of the allocation table. Then, select a zero cell having minimum original cost (say man M_i) and cross off all other zeros that correspond to job J_j and factory F_k. If more than one occurs, select any one.

Step 10: Reform the reduced table after removing fully used men, fully used job, and fully used factory.

Step 11: Repeat Step 9 and Step 10 until all men's are fully used, all jobs are fully used and all factories are fully used.

Step 12: This allotment yields an optimal solution to the given SAP (by Theorem 1).

4. Numerical Example:

Now, the reduction method is illustrated by the following example.

Example 1: Suppose that there are three men denoted by M_1, M_2 and M_3, three factories denoted by F_1, F_2 and F_3, and three jobs denoted by J_1, J_2 and J_3. It is known that c_{ijk} is the cost of the assigning job j to be performed by the man i in the factory k. Besides, three men, three factories and three jobs can be associated with only one of the others, that is, only one man with only one factory with only one job. The assignment costs c_{ijk} are given in the following table.

Factories	F_1			F_1			F_1		
		F_2			F_2			F_2	
			F_3			F_3			F_3
$Mens\downarrow/Jobs\rightarrow$	J_1			J_2			J_3		
M_1	10	8	12	9	10	27	15	10	13
M_2	8	6	7	9	6	12	7	11	12
M_3	9	7	6	10	7	12	8	6	8

Note that the given SAP is a balanced one.

Now, using the Step 2. to the Step 5. of the reduction method, we have the following M–J table.

	J_1			J_2			J_3		
	F_1	F_2	F_3	F_1	F_2	F_3	F_1	F_2	F_3
M_1	0	0	2	1	0	16	4	1	2
M_2	2	2	1	2	1	5	0	6	5
M_3	3	3	0	3	2	5	1	1	1

Now, using the Step 6. of the reduction method, we have the following J–F table.

	F_1			F_2			F_3		
	M_1	M_2	M_3	M_1	M_2	M_3	M_1	M_2	M_3
J_1	0	2	3	0	2	3	2	1	0
J_2	1	2	3	0	1	2	16	5	5
J_3	4	0	1	1	6	1	2	5	1

Now, using the Step 7. of the reduction method , we have the following F–M table.

	M_1			M_2			M_3		
	J_1	J_2	J_3	J_1	J_2	J_3	J_1	J_2	J_3
F_1	0	1	4	2	2	0	3	3	1
F_2	0	0	1	2	1	6	3	2	1
F_3	2	16	2	1	5	5	0	5	1

Now, using the Step 8. to the Step 11. of the reduction method, we obtain the following optimum allotment table.

Factories	F_1			F_1			F_1		
		F_2			F_2			F_2	
			F_3			F_3			F_3
$Mens \downarrow / Jobs \rightarrow$	J_1			J_2			J_3		
M_1	0	0	2	1	0 (10)	16	4	1	2
M_2	2	2	1	2	1	5	0 (7)	6	5
M_3	3	3	0 (6)	3	2	5	1	1	1

Therefore, the optimal solution to the given solid assignment problem is

$$M_3 \xrightarrow{F_3} J_1 \;,\; M_2 \xrightarrow{F_1} J_3 \text{ and } M_1 \xrightarrow{F_2} J_2$$

and the total minimum assignment cost is 23.

5. Conclusion

In this paper, we consider the three dimensional assignment problems. A new method namely reduction method for finding an optimal solution to solid assignment problems is proposed . The proposed method is easy to understand and to compute which is not based on the Hungarian method. The solution procedure of the reduction method is illustrated with help of a real life example. The proposed method can help decision makers in the three dimensional assignment related issues of real life problems by aiding them in the decision making process and providing an optimal solution in a simple and effective manner.

References

[1] E.Balas, M.J.Saltzman, An algorithm for the three-index assignment problem, Oper. Research, 39, (1991) 150–161.

[2] Crama, Y., Spieksma, F. C. R.: Approximation algorithms for three dimensional assignment problems with triangle inequalities. European Journal of Operational Research, 60, (1992), 273–279.

[3] Federico Perea, Multidimensional assignment: applications and some thoughts, Bolet´ın de Estad´ıstica e Investigaci´on Operativa, 27, (2011), 14–28.

[4] A.M.Frieze, L.Yadegar, An algorithm for solving 3-dimensional assignment problems with application to scheduling in a teaching practice, Journal Oper.Res.Soc.32, (1981) 989–995.

[5] D.Magos, Tabu search for the planar three-index assignment problem, J.Global Optim. 8, (1996) 35–48.

[6] W.P.Pierskalla, The multidimensional assignment problem, Oper. Research, 16, (1968), 422–431.

[7] A.B. Poore, Multidimensional assignment formulation of data association problems arising from multitarget and multisensor tracking, Comput. Optim. Appl., 3, (1994), 27– 54.

[8] A. B. Poore and A. J. Robertson III, A new Lagrangean relaxation based algorithm for a class of multidimensional assignment problems, Computational Optimization and Applications 8, (1997), 129–150.

[9] P. Storms, F. Spieksma, Geometric three-dimensional assignment problems.Comput. Oper. Res. 7, (2003), 1067–1085.

[10] Yusuke Kuroki and Tomomi Matsui, Approximation algorithm for multidimensional assignment problem minimizing the sum of squared errors, Mathematical Engg. Technical Reports, (2007), 1–20.