Minimizing Total Completion Time in a Two-machine Flow-shop Scheduling Problems

SHI Ling[1], CHENG Xue-guang[2]

(1Department of Mathematics, Hubei University for Nationalities, Enshi 445000, China,
2. School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China)

Abstract: We consider the problem of two-machine flow-shop scheduling with a single server and unit processing times, we show that this problem is NP-hard in the strong sense and present a busy schedule for it with worst-case bound $7/6$.

Key words: flow-shop scheduling problem, total completion time， worst-case, single server
Mathematics Subject Classification: 90B35

1 Introduction

We consider the two-machine flow-shop scheduling problem, which is described as follows. We are given n jobs $J_1, J_2, ..., J_n$, and two machines M_1 and M_2.Each job J_j consists of a chain $(O_{1,j}, O_{2,j}, ..., O_{n,j})$ of operations, and $O_{i,j}$ is to be processed on machine M_i for $p_{i,j}$ time units. Each machine can only process one operation at a time, and each job can be processed on at most one machine at a time. No preemption is allowed, i.e., once start, any operation can not be interrupted before it is completed. Immediately before processing an operation $O_{i,j}$ the corresponding machine, which takes a setup time of $s_{i,j}$ time units. During such a setup the machine is also occupied for $s_{i,j}$ time united, i.e., No other job can be processed on it. The setup times are assumed to be separable from the processing times, i.e., a setup on a subsequent machine may be performed while the job is still processed on the preceding machine. All setups have to be done by a single server M_S, which can perform at most one setup at a time. The problem we consider is to find a schedule S which minimizes the total completion times, that is $\sum_{j=1}^{n} C_j$. Following the three-field notation schedule introduced by Lentra et al [1], we denote this problem as $F2, S1\| \sum_{j=1}^{n} C_j$. If all processing are equal to 1, that is $p_{i,j} = 1\ (i = 1,2, j = 1,2,...,n)$, we have the $F2, S1\| \sum_{j=1}^{n} C_j$ problem.

Complexity results for flow-shop problems obtained by Garey, et al [2], who studied two-machine flow-shop problem with minimizing total completion times, that is $F2\| \sum_{j=1}^{n} C_j$. J.A.Hoogereen, et al [3] studied some special cases for two-machine flow-shop problems with minimizing total completion times, and proved that the problem with equal processing on first machine, that is $F2,S1|p_{i,j}=1|\sum_{j=1}^{n} C_j$,is NP -hard in the strong sense, and present an $O(n\log n)$ approximation algorithm for it with worst-case bound $4/3$ .Complexity results for flow-shop problems with a single server was obtained by Brucher, et al [4]. The complexity of parallel dedicated machine with a single server was obtained by Glass, et al [5].

2 Complexity of the $F2,S1|p_{i,j}=1|\sum_{j=1}^{n} C_j$ problem

Let $C_{i,j}$ denote the completion times of job J_j on machine M_i. If there are no idle times on machine M_1 and machine M_2, we have

$C_{1,1}=s_{1,1}+p_{1,1}$, $C_{2,1}=s_{1,1}+p_{1,1}+s_{2,1}+p_{2,1}$, $C_{1,j}=C_{1,j-1}+s_{1,j}+p_{1,j}$,

$C_{2,j}=\max\{C_{2,j-1},C_{1,j}\}+s_{2,j}+p_{2,j}$, for $j=2,3,\ldots,n$.

Theorem1: The $F2,S1|p_{i,j}=1|\sum_{j=1}^{n} C_j$ problem is NP -hard in the strong sense.

Proof Our proof is based upon a reduction from the problem Numerical Matching with Target Sums or, in short, Target Sum, which is known to be NP -hard in the strong sense[6].

Target Sum. Given two multisets $X=\{x_1,x_2,\ldots,x_n\}$ and $Y=\{y_1,.y_2,\ldots,y_n\}$ of positive integers and an target vector $\{z_1,.z_2,\ldots,z_n\}$,where $\sum_{j=1}^{n}(x_j+y_j)=\sum_{j=1}^{n} z_j$,is there a position of the set $X\cup Y$ into n disjoint set $Z_1,Z_2,\ldots,Z_n$,each containing exactly one element from each of X and Y ,such that the sum of the numbers in Z_j equal z_j ,for $1,2,\ldots,n$?

(1) P -job: $s_{1,i}=1, p_{1,i}=1; s_{2,i}=b+x_i, p_{2,i}=1(i=1,2,\ldots,n)$,

(2) Q -job: $s_{1,i}=0, p_{1,i}=1; s_{2,i}=b+y_i, p_{2,i}=1(i=1,2,\ldots,n)$,

(3) R -job: $s_{1,i}=0, p_{1,i}=1; s_{2,i}=b-z_i, p_{2,i}=1(i=1,2,\ldots,n)$,

(4) U -job: $s_{1,i}=0, p_{1,i}=1; s_{2,i}=0, p_{2,i}=1(i=1,2,\ldots,(3b-1)n)$,

(5) L -job: $s_{1,i}=3b, p_{1,i}=1; s_{2,i}=1, p_{2,i}=1(i=1,2,\ldots,n)$.

Observe that all processing times are equal to 1.To prove the theorem we show that in this constructed if the $F2,S1|p_{i,j}=1|\sum_{j=1}^{n}C_j$ problem a schedule S_0 satisfying $\sum_{j=1}^{n}C_j(S_0)\le y=\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}(x_i+y_i)+(77n^2-13n-4)b/2$ exists if and only if Target Num has a solution.

Suppose that Target Num has a solution. The desired schedule S_0 exists and can be described as follows. No machine has intermediate idle time. Machine M_1 process the jobs in order of the sequence σ, i.e., in the sequence

$$\sigma=\{\sigma_{P_{1,1}},\sigma_{Q_{1,1}},\sigma_{R_{1,1}},\sigma_{U_{1,1}},\sigma_{V_{1,1}},\sigma_{W_{1,1}},\sigma_{L_{1,1}},\ldots,\sigma_{P_{1,n}},\sigma_{Q_{1,n}},\sigma_{R_{1,n}},\sigma_{U_{1,n}},\sigma_{V_{1,n}},\sigma_{W_{1,n}}$$

While machine M_2 process the jobs in the sequence

$$\tau=\{\tau_{P_{2,1}},\tau_{Q_{2,1}},\tau_{R_{2,1}},\tau_{U_{2,1}},\tau_{V_{2,1}},\tau_{W_{2,1}},\tau_{L_{2,1}},\ldots,\tau_{P_{2,n}},\tau_{Q_{2,n}},\tau_{R_{2,n}},\tau_{U_{2,n}},\tau_{V_{2,n}},\tau_{W_{2,n}},\tau_{L_{2,n}}\}$$

as indicated in Figure 1.

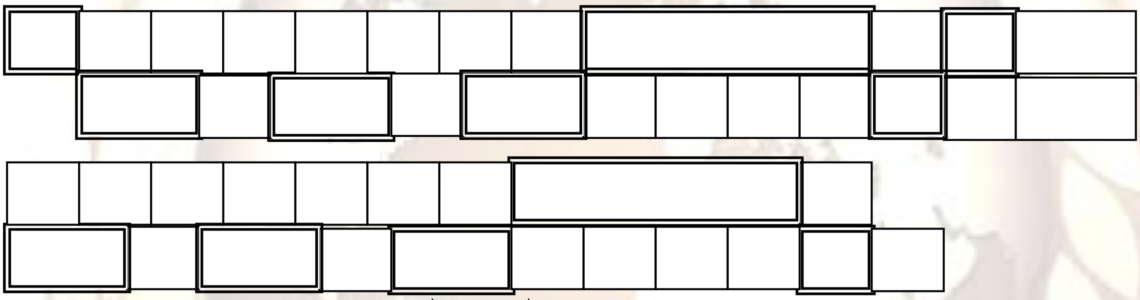

Fig.1 Gant chart for the $F2,S1|p_{i,j}=1|\sum_{j=1}^{n}C_j$ problem

Then we define the sequence σ and τ shown in Figure 1. Obviously, these sequence σ and τ fulfills $C(S)=C(\sigma,\tau)\le y$. Conversely, assume that the flow-shop scheduling problem has a solution σ and τ with $C(S)\le y$.

Considering the path composed of machine M_1 operations of jobs $\{P_{1,1},Q_{1,1},R_{1,1},U_{1,1},V_{1,1},W_{1,1}\}$, Machine M_2 operations of jobs $\{R_{2,1},U_{2,1},V_{2,1},W_{2,1},L_{2,1},\ldots,R_{2,n},U_{2,n},V_{2,n},W_{2,n},L_{2,n}\}$, we obtain that

$$C(S)\ge 3b+x_1+5b+x_1+y_1+7b+x_1+y_1-z_1+8b+9b+10b+\ldots+$$

$$(3+(n-1)11)b+x_+ +(5+(n-1)11)b+x_n+y_n+(7+(n-1)11)b+\ldots+(11n+1)b=$$

$$\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}(x_i+y_i)+(77n^2-13n-4)b/2=y,$$ So $C(S)=y$.

(a) If S has a partition μ ,then there is a schedule with finish times y. One such schedule is shown in Figure 1.

(b) If S has no partition, then all schedule must have a finish times $> y$. Since S has no partition, then $x_i + y_i \neq z_i (i = 1,2,...,n)$. Let $\xi_i = x_i + y_i - z_i (i = 1,2,...,n)$, we have

$$\sum C_j(S) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} (x_i + y_i) + (77n^2 - 31n - 2)b/2 + 5\sum_{i=1}^{n} \xi_i + 10\sum_{i=1}^{n-1} \xi_i + ... + 5n\xi_1 > y$$

3 Worst-case for the $F2, S1 \| p_{i,j} = 1 \| \sum_{j=1}^{n} C_j$ problem

In examining "worst" schedule, we restrict ourselves to busy schedule. A busy schedule is a schedule in which at all times from start to finish at least one server is processing a task.

Theorem 2 The $F2, S1 \| p_{i,j} = 1 \| \sum_{j=1}^{n} C_j$ problem, let S_0 be a busy schedule for this problem, S^* be the optimal solution for the $F2, S1 \| p_{i,j} = 1 \| \sum_{j=1}^{n} C_j$ problem then $\sum_{j=1}^{n} C_j(S_0) / \sum_{j=1}^{n} C_j(S^*) \leq 7/6$, The bound is tight.

Proof For a schedule S, let $I_{i,j}(S)(i = 1,2; j = 1,2,...,n)$ denote the total idle times of job J_j on machine M_i.

Considering the path composed of machine M_1 operations of jobs $1, 2, ..., j$, machine M_2 operation of job j, we obtain that $C_j = \sum_{i=1}^{j} (s_{1,i} + p_{1,i}) + I_{1,j} + s_{2,j} + p_{2,j}$ (1)

Considering the path composed of machine M_1 operations of jobs 1, machine M_2 operation of job $1, 2, ..., j$, we obtain that $C_j = s_{1,1} + p_{1,1} + \sum_{i=1}^{j} (s_{2,i} + p_{2,i}) + I_{2,j}$ (2)

Considering the path composed of machine M_1 operations of jobs $1, 2, ..., l$, machine M_2 operation of job $l, l+1, ..., j$, we obtain that

$$C_j = \sum_{i=1}^{l} (s_{1,i} + p_{1,i}) + I_{1,l} + \sum_{l=1}^{j} (s_{2,i} + p_{2,i}) + I_{2,j} \quad (3)$$

So we have

$$6\sum_{j=1}^{n} C_j(S_0) = 2(\sum_{i=1}^{j} (s_{1,i} + p_{1,i}) + I_{1,j} + s_{2,j} + p_{2,j}) + 2(s_{1,1} + p_{1,1} + \sum_{i=1}^{j} (s_{2,i} + p_{2,i} + I_{2,j})) +$$
$$\sum_{i=1}^{l} (s_{1,i} + p_{1,i}) + I_{1,l} + \sum_{l=1}^{j} (s_{2,i} + p_{2,i}) + I_{2,j}$$

$$= 2(\sum_{i=1}^{j} (s_{1,i} + p_{1,i}) + I_{1,j}) + 2(\sum_{i=1}^{j} (s_{2,i} + p_{2,i}) + I_{2,j}) + 2(\sum_{i=1}^{l} (s_{1,i} + p_{1,i}) + I_{1,l} +$$

$$(\sum_{i=1}^{j}(s_{2,i}+p_{2,i})+I_{2,j})+(2(s_{11}+p_{11})+2(s_{2,j}+p_{2,j}))\le 7\sum_{j=1}^{n}C_j(S^*),$$

$$\sum_{j=1}^{n}C_j(S_0)/\sum_{j=1}^{n}C_j(S^*)\le 7/6.$$

To prove the bound is tight, introduce the following example as show in Fig.2 and Fig. 3.

(1) $s_{1,i}=2, p_{1,i}=1, s_{2,i}=1, p_{2,i}=1 (i=1,2)$; (2) $s_{1,i}=0, p_{1,i}=1, s_{2,i}=0, p_{2,i}=1 (i=1,2)$

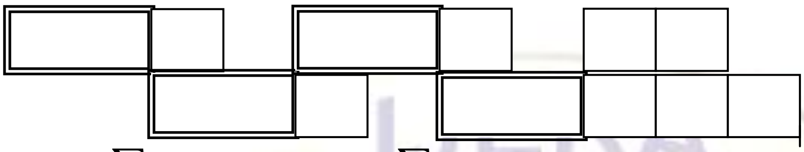

Fig.2 $\sum C_j(S_0)$ $\sum C_j(S_0)$ =35

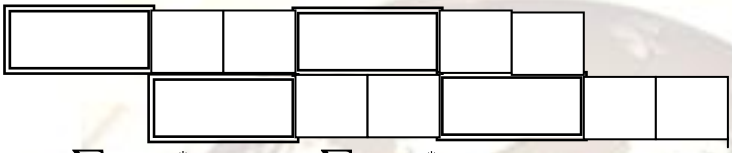

Fig.3 $\sum C_j(S^*)$ $\sum C_j(S^*)$ =30

So we have $\sum_{j=1}^{n}C_j(S_0)/\sum_{j=1}^{n}C_j(S^*)=35/30=7/6$, the bound is tight.

References

[1] E.L. Lawer, J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys. Sequencing and Scheduling: Algorithms and Complexity, in: S.C. Gtaves, A.H.G. Rinnooy Kan,P.H. zipkin(Eds.), Handbooks in Operation Research and Management Science, Vol.4, Logistics of Production and Inventory, Northhollland, Amsterrdem, 1993,pp.445~522.

[2] M.R.Garey, D.S.Johson, and R.Sethi. The complexity of flowshop and jobshop scheduling. Math. Oper.Res.m, (12) 1976:117-129。

[3] J.A. Hoogereen, T.Kaweguchi. Minimizing total completion time in a two-machine flowshop: analysis of special cases. Processing of the 5th IPCO conference, 1996:374-388.

[4] P. Brucker, S. Knust,G.Q. Wang,et al. Complexity of results for flow-shop problems with a single server. European J. Oper. Res., 165(2)， 2005:398-407.

[5] C.A. Glass, Y.M.Shafrausky, V.A.Strusevich. Scheduling for parallel dedicated machines with a single server.Naval Research Logistics, 2000, 47:304-328.

[6] M.R.Garey,D.S.Johnson. Computers and Intractability: A Guide to the Theory of NP -completeness ,Freman, San Francisco.

1 SHI Ling（author）, Department of Mathematics, Hubei University for Nationalities, Enshi 445000,
2 CHENG Xue-guang（corresponding author）, School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China,