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ON SOME MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract:

In the present paper, we derive three new composition formulae of a class of multidimensional fractional integral operators involving generalized multivariable polynomial and I-function. On account of the general nature of the functions occurring as kernels here, the main findings of our paper are capable of yielding a number of corresponding results (new and known) involving simpler functions and polynomials (of one or more variables) as special cases of our formulae. We also give a two dimensional analogue of our second composition formulae. The results obtained by Erdelyi [1], Goyal and Jain [2], Goyal, Jain and Gaur [3], follow as simple cases of our composition formulae.

Key Words: Fractional Integral Operators, I-function, Mellin Transform, Stieltjes Transform, General Class of Multivariable Polynomials.
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1. Introduction

The subject of fractional calculus (that is, calculus of derivatives and integrals of an arbitrary order) has gained considerable importance and popularity during the past four decades or so, due mainly to its applications in numerous diverse fields of science and engineering. Fractional calculus is applicable in deriving the solutions of certain integral equations involving special functions of Mathematical physics which possess a Mellin-Barnes type integral representation.

In recent years several authors (see, for example) Erdelyi [1], Nishimoto [5,6], Singh and Mandia [10] have made significant contributions to the fractional calculus operators involving various functions and polynomials. Srivastava and Saxena [12] have presented a systematic account of Fractional calculus operators and their applications investigated by various authors. In the present paper we introduce and study a new pair of Fractional integral operators defined and represented in the following manner:

$$I_x\left[f(t_1,\dots,t_s)\right]=I_{x,u,v,z}^{\rho,\sigma;e,f;\eta,\lambda}\left[f(t_1,\dots,t_s);x_1,\dots,x_s\right]=\left(\prod_{j=1}^{s}x_j^{-\rho_j-\sigma_j}\right)$$

$$\int_0^{x_1}\dots\int_0^{x_s}\left[\prod_{j=1}^{s}t_j^{\rho_j}(x_j-t_j)^{\sigma_j-1}\right]S_V^{U_1,\dots,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},\dots,E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right]$$

$$I_{p_i,q_i:r}^{m,n}\left[Z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j}\middle|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right]f(t_1,\dots,t_s)dt_1\dots dt_s \tag{1.1}$$

$$J_x\left[f(t_1,\dots,t_s)\right]=J_{x,u,v,z}^{\rho,\sigma;e,f;\eta,\lambda}\left[f(t_1,\dots,t_s);x_1,\dots,x_s\right]=\left(\prod_{j=1}^{s}x_j^{\rho_j}\right)$$

$$\int_{x_1}^{\infty}\dots\int_{x_s}^{\infty}\left[\prod_{j=1}^{s}t_j^{-\rho_j-\sigma_j}(t_j-x_j)^{\sigma_j-1}\right]S_V^{U_1,\dots,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{e_1}\left(1-\frac{x_1}{t_1}\right)^{f_1},\dots,E_s\left(\frac{x_s}{t_s}\right)^{e_s}\left(1-\frac{x_s}{t_s}\right)^{f_s}\right]$$

$$I_{p_i,q_i:r}^{m,n}\left[Z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{\eta_j}\left(1-\frac{x_j}{t_j}\right)^{\lambda_j}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]f(t_1,...,t_s)dt_1...dt_s \qquad (1.2)$$

Throughout the paper, we assume that

$$f(t_1,...,t_s)=\begin{cases}O\prod_{j=1}^{s}\left(|t_j|^{U_j}\right), & \max\{|t_j|\}\to 0\\ O\prod_{j=1}^{s}\left(|t_j|^{-V_j}e^{-W_j|t_j|}\right), & \min\{|t_j|\}\to\infty\end{cases} \quad j=1,...,s \qquad (1.3)$$

Such a class of function will be represented symbolically as $f(t_1,...,t_s)\in A$.

We also assume that $\int ... \int_{\Omega_s} |f(t_1,...,t_s)|\,dt_1...dt_s<\infty$ for every bounded s-dimensional region Ω_s excluding the region. The operators defined by (1.1) and (1.2) exist if

(i) $\min \mathrm{Re}(e_j, f_j, \eta_j, \lambda_j)\geq 0 \quad (j=1,...,s)$ not all zero simultaneously;

(ii) $\min\limits_{1\leq k\leq m}\mathrm{Re}\left[1+\rho_j+U_j+\eta_j\dfrac{b_j}{\beta_j}\right]>0, \min\limits_{1\leq k\leq m}\left[\sigma_j+\lambda_j\dfrac{b_j}{\beta_j}\right]>0;$

(iii) $\mathrm{Re}(W_j)=0, \min\limits_{1\leq k\leq m}\mathrm{Re}\left[\rho_j+V_j+\eta_j\dfrac{b_k}{\beta_k}\right]>0,\ \min\limits_{1\leq k\leq m}\mathrm{Re}\left[\sigma_j+\lambda_j\dfrac{b_k}{\beta_k}\right]>0;$

Or $\mathrm{Re}(W_j)>0,\ \min\limits_{1\leq k\leq m}\mathrm{Re}\left[\sigma_j+\lambda_j\dfrac{b_k}{\beta_k}\right]>0; \qquad (1.4)$

The multivariable polynomial $S_V^{U_1,...,U_k}(x_1,...,x_k)$ introduced by Srivastava and Garg [11,p.686, eq. (1.4)} is defined in the following manner:

$$S_V^{U_1,...,U_k}[x_1,...,x_k]=\sum_{R_1,...,R_k=0}^{\sum_{i=1}^{k}U_iR_i\leq V}(-V)_{\sum_{i=1}^{k}U_iR_i}A(V,R_1,...,R_k)\frac{x_i^{R_i}}{R_i!},\ V=0,1,2,... \qquad (1.5)$$

Where $U_1,...,U_k$ are arbitrary positive integers and the coefficients $A(V,R_1,...,R_k)$ are arbitrary constants (real or complex).

The I - function occurring in the paper was introduced by Saxena [9] defined in the following form:

$$I[z]=I_{p_i,q_i:r}^{m,n}[z]=I_{p_i,qi;r}^{m,n}\left[z\left|\begin{matrix}\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}\\\{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L t(s)z^s ds \qquad (1.6)$$

Where $\omega=\sqrt{-1}$;

$$t(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s\right\}} \qquad (1.7)$$

$p_i,q_i(i=1,....,r)$,m,n are integers satisfying $0\leq n\leq p_i; 0\leq m\leq q_i \quad (i=1,....,r)$, r is finite, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and a_j,b_j,a_{ji},b_{ji} are complex numbers such that

$\alpha_j(b_h+\upsilon)\neq\beta_h(a_j-\upsilon-k)$, for υ, k =1, 2, ...; h = 1, 2, . . . , m; j = 1, 2, . . . , n. L is a contour which runs from $\sigma-w^{\infty}$ to $\sigma+w^{\infty}$ (σ is real),

$$s=(a_j-1-\upsilon)/\alpha_j;\ j=1,2,\ldots,n;\ \upsilon=0,1,2,\ldots$$

$$s=(b_j+\upsilon)/\beta_j;\ j=1,2,\ldots,m;\ \upsilon=0,1,2,\ldots$$

Lie to the L.H.S. and R.H.S. of L, respectively.

The following sufficient conditions for the absolute convergence of the defining integral for I -function given by (1.6) are

$$|\arg z|<\frac{1}{2}\pi\Omega, \tag{1.8}$$

Where

$$\Omega=\sum_{j=1}^{m}\beta_j+\sum_{j=1}^{n}\alpha_j-\sum_{j=m+1}^{q_i}\beta_{ji}-\sum_{j=n+1}^{p_i}\alpha_{ji}>0. \tag{1.9}$$

The following series representation of the I -function was given as:

$$I_{p_i,q_i:r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\sum_{v=1}^{m}\sum_{p=0}^{\infty}\theta(S_{p,v})z^{S_{p,v}} \tag{1.10}$$

Where

$$\theta(S_{p,v})=\frac{\prod_{j=1,j\neq v}^{m}\Gamma\left(b_j-\beta_jS_{p,v}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_jS_{p,v}\right)(-1)^p}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}+\beta_{ji}S_{p,v}\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_j-\alpha_jS_{p,v}\right)\right\}p!\beta_v},S_{p,v}=\frac{b_v+p}{\beta_v}$$

In the sequel, we shall also make use of the following behavior of the I -function for small and large value of z as recorded by Saxena [9]:

$$I_{p_i,q_i:r}^{m,n}[z]=O[|z|^{\alpha}]\text{ for small }z\text{, where }\alpha=\min_{1\leq j\leq m}\left[\operatorname{Re}\frac{b_j}{\beta_j}\right] \tag{1.12}$$

$$I_{p_i,q_i:r}^{m,n}[z]=O[|z|^{\beta}]\text{ for large }z\text{, where }\beta=\max_{1\leq j\leq n}\left[\operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right)\right] \tag{1.13}$$

And the conditions (1.8) and (1.9) are also satisfied.

2. Composition Formula for the Multidimensional Fractional Integral Operators

Result 1

$$I_{x;U,V;Z}^{\rho,\sigma;E,F;\eta,\lambda}\left\{J_{y;U',V';Z'}^{\rho',\sigma';E',F';\eta',\lambda'}\left[f(t_1,\ldots,t_s)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{\rho_j-1}\right)\int_0^{x_1}\ldots\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\rho_j}\right)G\left(\frac{t_1}{x_1},\ldots,\frac{t_s}{x_s}\right)f(t_1,\ldots,t_s)dt_1\ldots dt_s$$

$$+\left(\prod_{j=1}^{s}x_j^{\rho'_j-1}\right)\int_{x_1}^{\infty}\ldots\int_{x_s}^{\infty}\left(\prod_{j=1}^{s}t_j^{-\rho'_j-1}\right)G\left(\frac{x_1}{t_1},\ldots,\frac{x_s}{t_s}\right)f(t_1,\ldots,t_s)dt_1\ldots dt_s, \tag{2.1}$$

Where

$$G(t_1,\ldots,t_s)=\sum_{R_1,\ldots,R_s=0}^{\sum_{j=1}^{s}U_jR_j\leq V}(-V)_{\sum_{j=1}^{s}U_jR_j}A(V,R_1,\ldots,R_s)\frac{E_1^{R_1}}{R_1!}\ldots\frac{E_s^{R_s}}{R_s!}$$

$$\sum_{R'_1,\dots,R'_s=0}^{\sum_{j=1}^{s}U'_jR'_j\le V'} (-V')_{\sum_{j=1}^{s}U'_jR'_j} A(V',R'_1,\dots,R'_s)\frac{E'^{R'_1}_1}{R'_1!}\cdots\frac{E'^{R'_s}_s}{R'_s!}$$

$$\sum_{v=1}^{m'}\sum_{p'=0}^{\infty}\theta\left(S_{p',v}\right)z'^{S_{p',v}}\,\Gamma\left(\sigma'_j+f'_jR'_j+\lambda'_jS_{p',v}+l\right)t_j^{e'_jR'_j+l}$$

$$I^{m,n+2s}_{p_i+2s,q_i+2s:r}\left[z\prod_{j=1}^{s}(t_j)^{\eta_j}(1-t_j)^{\lambda_j}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right], \qquad (2.2)$$

Where

$$A^*=(a_k,\alpha_k)_{1,n}(-\rho_j-\rho'_j-e_jR_j-e'_jR'_j-\eta'_jS_{p',v},\eta_j)$$

$$(-\rho_j-\rho'_j-\sigma_j-\sigma'_j-l-1-(e_j+f_j)R_j+(e'_j+f'_j)R'_j+(\eta'_j+\lambda'_j)S_{p',v},\eta_j)(a_{ji},\alpha_{ji})_{n+1,p_i} \quad (2.3)$$

$$B^*=(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,q_i}(-\rho_j-\rho'_j-\sigma'_j-l-e_jR_j-(e'_j+f'_j)R'_j-(\eta'_j+\lambda'_j)S_{p',v},\eta'_j)$$

$$(-\rho_j-\rho'_j-\sigma_j-\sigma'_j-1-(e_j+f_j)R_j-(e'_j+f'_j)R'_j-(\eta'_j+\lambda'_j)S_{p',v}(\eta_j+\lambda_j)) \quad , \quad (2.4)$$

and $G'(t_1,\dots,t_s)$ can be obtained from $G(t_1,\dots,t_s)$ from (2.2) by interchanging the parameters with dashes with those without dashes, $f(t_1,\dots,t_s)\in A$, the composite operator defined by the L.H.S. of (2.1) exists and the following conditions are satisfied:

$$\min_{1\le k\le m}\mathrm{Re}\left[1+\rho_j+\eta_j\frac{b_k}{\beta_k}\right]>0,\ \min_{1\le k\le m}\mathrm{Re}\left[1+\rho'_j+U_j+\eta'_j\frac{b_k}{\beta_k}\right]>0,$$

$$\min_{1\le k\le m}\mathrm{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0,\ \min_{1\le k\le m}\mathrm{Re}\left[\sigma'_j+\lambda'_j\frac{b'_k}{\beta'_k}\right]>0, \text{ and}$$

$$\mathrm{Re}(w_j)>0 \ or\ \mathrm{Re}(w_j)=0,\ \min_{1\le k\le m}\mathrm{Re}\left[1+\rho_j+\sigma_j+V_j+\eta_j\frac{b_k}{\beta_k}\right]>0. \qquad (2.5)$$

Result 2

$$I^{\rho,\sigma;E,F;\eta,\lambda}_{x;U,V;Z}\left\{I^{\rho',\sigma';E',F';\eta',\lambda'}_{y;U',V';Z'}\left[f(t_1,\dots,t_s)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{-\rho'_j-1}\right)\int_0^{x_1}\!\!\dots\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\rho'_j}\right)G\left(\frac{t_1}{x_1},\dots,\frac{t_s}{x_s}\right)f(t_1,\dots,t_s)dt_1\dots dt_s\ , \qquad (2.6)$$

Where

$$G(t_1,\dots,t_s)=\sum_{R_1,\dots,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}(-V)_{\sum_{j=1}^{s}U_jR_j}A(V,R_1,\dots,R_s)\frac{E_1^{R_1}}{R_1!}\cdots\frac{E_s^{R_s}}{R_s!}$$

$$\sum_{R'_1,\dots,R'_s=0}^{\sum_{j=1}^{s}U'_jR'_j\le V'} (-V')_{\sum_{j=1}^{s}U'_jR'_j} A(V',R'_1,\dots,R'_s)\frac{E'^{R'_1}_1}{R'_1!}\cdots\frac{E'^{R'_s}_s}{R'_s!}$$

$$\sum_{v=1}^{m'}\sum_{p'=0}^{\infty}\theta\left(S_{p',v}\right)z'^{S_{p',v}}\,\Gamma\left(\sigma'_j+f'_jR'_j+\lambda'_jS_{p',v}+l\right)t_j^{e'_jR'_j+l}$$

$$I^{m,n+2s}_{p_i+2s,q_i+2s:r}\left[z\prod_{j=1}^{s}(t_j)^{\eta_j}(1-t_j)^{\lambda_j}\left|\begin{matrix}A^{**}\\B^{**}\end{matrix}\right.\right] \qquad (2.7)$$

Where

$$A^{**} = (a_k, \alpha_k)_{1,n}, (a_{ki}, \alpha_{ki})_{n+1,p_i}, (-\rho_j - \rho_j^{'} - e_j R_j - e_j^{'} R_j^{'} - \eta_j^{'} S_{p',v}, \eta_j)$$
$$(-\rho_j + \rho_j^{'} + \sigma_j^{'} - e_j R_j + (e'_j + f'_j) R'_j + (\eta_j^{'} + \lambda_j^{'}) S_{p',v}, \eta_j) \quad (2.8)$$
$$B^{**} = (b_j, \beta_j)_{1,m} (b_{ji}, \beta_{ji})_{m+1,q_i} (1 - l - \rho_j - \sigma_j - f_j R_j - f'_j R'_j - \lambda'_j S_{p',v}, \lambda_j)$$
$$(-\rho_j - \rho'_j - \sigma_j - \sigma_j^{'} - 1 - (e_j + f_j) R_j - (e'_j + f'_j) R'_j - (\eta'_j + \lambda'_j) S_{p',v} (\eta_j + \lambda_j)) \quad (2.9)$$

Where $\theta(S_{p,v})$ and $S_{p,v}$ are given by (1.13) and the following conditions are satisfied:

$$\min_{1\le k\le m} \operatorname{Re}\left[1 + \rho_j + \eta_j \frac{b_k}{\beta_k}\right] > 0, \min_{1\le k\le m} \operatorname{Re}\left[1 + \rho'_j + U_j + \eta'_j \frac{b_k}{\beta_k}\right] > 0,$$

$$\operatorname{Re}(w_j) > 0 \ or \ \operatorname{Re}(w_j) = 0, \min_{1\le k\le m} \operatorname{Re}\left[1 + \rho_j + \sigma_j + V_j + \eta_j \frac{b_k}{\beta_k}\right] > 0 \quad (2.10)$$

Result 3

$$J_{x;U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'} \left\{ J_{y;U,V,Z}^{\rho,\sigma;e,f;\eta,\lambda} \left[f(t_1,...,t_s) \right] \right\}$$
$$= \left(\prod_{j=1}^{s} x_j^{\rho'_j} \right) \int_{x_1}^{\infty} ... \int_{x_s}^{\infty} \left(\prod_{j=1}^{s} t_j^{-\rho'_j - 1} \right) G\left(\frac{x_1}{t_1}, ..., \frac{x_s}{t_s} \right) f(t_1,...,t_s) dt_1 ... dt_s, \quad (2.11)$$

Where $G(t_1,...,t_s)$ is given by (2.7), $f(t_1,...,t_s) \in A$, the composite operator defined by the L.H.S. of (2.11) exists and the following conditions are satisfied:

$$\min_{1\le k\le m} \operatorname{Re}\left[\sigma_j + \lambda_j \frac{b_k}{\beta_k}\right] > 0, \min_{1\le k\le m} \operatorname{Re}\left[\sigma'_j + \lambda'_j \frac{b'_k}{\beta'_k}\right] > 0, \text{ and}$$

$$\operatorname{Re}(w_j) > 0 \ or \ \operatorname{Re}(w_j) = 0, \min_{1\le k\le m} \operatorname{Re}\left[1 + \rho_j + \sigma_j + V_j + \eta_j \frac{b_k}{\beta_k}\right] > 0 \quad (2.12)$$

Proof: To prove result 1, we first express both the I - and J -operators involved in the L.H.S. of (2.1), in the integral form with the help of (1.1) and (1.2). Then we interchange the order of t_j - and y_j - integral (which is permissible under the conditions stated) and get the following integral:

$$I_{x;U,V;Z}^{\rho,\sigma;E,F;\eta,\lambda} \left\{ J_{y;U',V';Z'}^{\rho',\sigma';E',F';\eta',\lambda'} \left[f(t_1,...,t_s) \right] \right\}$$
$$= \int_0^{x_1} ... \int_0^{x_s} \left\{ \int_0^{t_1} ... \int_0^{t_s} \Omega dy_1 .... dy_s \right\} f(t_1,...,t_s) dt_1 ... dt_s + \int_{x_1}^{\infty} ... \int_{x_s}^{\infty} \left\{ \int_0^{x_1} ... \int_0^{x_s} \Omega dy_1 .... dy_s \right\} f(t_1,...,t_s) dt_1 ... dt_s$$
$$= \int_0^{x_1} ... \int_0^{x_s} I_1 f(t_1,...t_s) dt_1 ... dt_s + \int_{x_1}^{\infty} ... \int_{x_s}^{\infty} I_2 f(t_1,...,t_s) dt_1 ... dt_s, \quad (2.13)$$

Where

$$\Omega = \left(\prod_{j=1}^{s} x_j^{-\rho_j - \sigma_j} t_j^{-\rho_j^{'} - \sigma_j^{'}} y_j^{\rho_j + \rho_j^{'}} (x_j - y_j)^{\sigma_j - 1} (t_j - y_j)^{\sigma_j^{'} - 1} \right)$$
$$S_V^{U_1,...,U_s} \left[E_1 \left(\frac{y_1}{x_1} \right)^{e_1} \left(1 - \frac{y_1}{x_1} \right)^{f_1}, ..., E_s \left(\frac{y_s}{x_s} \right)^{e_s} \left(1 - \frac{y_s}{x_s} \right)^{f_s} \right] S_{V'}^{U'_1,...,U'_s} \left[E'_1 \left(\frac{y_1}{t_1} \right)^{e_1} \left(1 - \frac{y_1}{t_1} \right)^{f_1}, ..., E'_s \left(\frac{y_s}{t_s} \right)^{e_s} \left(1 - \frac{y_s}{t_s} \right)^{f_s} \right]$$
$$I_{p_i,q_i:r}^{m,n} \left[z \prod_{j=1}^{s} \left(\frac{y_j}{x_j} \right)^{\eta_j} \left(1 - \frac{y_j}{x_j} \right)^{\lambda_j} \right] I_{p'_i,q'_i:r}^{m',n'} \left[z' \prod_{j=1}^{s} \left(\frac{y_j}{t_j} \right)^{\eta'_j} \left(1 - \frac{y_j}{t_j} \right)^{\lambda'_j} \right] dy_1 ... dy_s \quad (2.14)$$

To evaluate I_1, involved in the first integral on the R.H.S. of (2.13), we express both the multivariable polynomials $S_V^{U_1,...,Us}, S_{V'}^{U'_1,...,U'_s}$ and $I_{p'_i,q'_i:r}^{m',n'}$ in terms of their respective series with the help of equations (1.5) and (1.12) respectively, $I_{p'_i,q'_i:r}^{m',n'}$ is expressed in terms of the Mellin-Barnes contour integral with the help of (1.6). Now interchanging the order of summations and Mellin-Banes contour integral with –integral and further, evaluating the –integral by setting in (2.14) and using the known result [8,p.47, Th. 1.6], we get the following form after a little simplification:

$$I_1 = \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s} U_j R_j \le V} (-V)_{\sum_{j=1}^{s} U_j R_j} A(V, R_1,..., R_s) \frac{E_1^{R_1}}{R_1!} \cdots \frac{E_s^{R_s}}{R_s!}$$

$$\sum_{R'_1,...,R'_s=0}^{\sum_{j=1}^{s} U'_j R'_j \le V'} (-V')_{\sum_{j=1}^{s} U'_j R'_j} A(V', R'_1,..., R'_s) \frac{E'^{R'_1}_1}{R'_1!} \cdots \frac{E'^{R'_s}_s}{R'_s!}$$

$$\sum_{v=1}^{m'} \sum_{p'=0}^{\infty} \theta(S_{p',v}) z'^{S_{p',v}} \frac{1}{2\pi i} \int_L \phi(\xi) Z^{\xi} d\xi\, t_j^{\rho_j + e_j R_j + \eta_j \xi} x_j^{-\rho_j - e_j R_j - \eta_j \xi - 1}$$

$$\frac{\Gamma\left(\rho_j + \rho'_j + e_j R_j + e' R'_j + \eta'_j S_{p',v} + \eta_j \xi + 1\right) \Gamma\left(\sigma'_j + f'_j R'_J + \lambda'_j S_{p',v}\right)}{\Gamma\left(\rho_j + \rho'_j + \sigma'_j + e_j R_j + (e' + f'_j) R'_j + (\eta'_j + \lambda'_j) S_{p',v} + 1\right)}$$

$${}_2F_1\left[\begin{matrix} 1+\rho_j+\rho'_j+\sigma'_j+e_jR_j+e'_jR'_j+\eta'_jS_{p',v}+\eta_j\xi, 1-\sigma_j+f_jR_j+\lambda_j\xi \\ 1+\rho_j+\rho'_j+\sigma'_j+e_jR_j+(e'_j+f'_j)R'_j+(\eta'_j+\lambda'_j)S_{p',v}+\eta_j\xi \end{matrix}; \frac{1}{x}\right].$$

Now applying the transformation formula [8,p.60, eq. (5)], expressing the thus obtained in the series form and re-interpreting the result in terms of I -function we get the solution of I_1.

To calculate I_2, we proceed on similar lines, with the difference that the $I_{p_i,q_i:r}^{m,n}$ -function is now expressed in series and another one is expressed in terms of Mellin-Barnes contour integral.

Result 2 and Result 3 can be proved similarly by setting $\frac{x_j - y_j}{x_j - t_j} = u_j$ and using the known result [4,p. 287, eq. 3.197(8)].

3. Special Cases

We now presenting a two dimensional analogue of our second composition formula by taking $s = 2$ after reducing the generalized class of polynomials to unity:

$$I_{x,y,Z}^{\rho,\sigma,\eta,\lambda} \left\{ I_{s,t,Z'}^{\rho',\sigma',\eta',\lambda'} [f(u,v)] \right\} = I_{x,y,Z}^{\rho,\sigma,\eta,\lambda} \left\{ s^{-\rho'-\sigma'} t^{-m'-n'} \int_0^s \int_0^t u^{\rho'} v^{m'} (s-u)^{\sigma'-1} (t-v)^{n'-1} \right.$$

$$\left. I_{p'_i,q'_i:r}^{m',n'} \left[z' \left(\frac{u}{s}\right)^{\eta'} \left(\frac{v}{t}\right)^{\delta'} \left(1-\frac{u}{s}\right)^{\lambda'} \left(1-\frac{v}{t}\right)^{\mu'} \right] f(u,v)\, du dv \right\}$$

$$= \sum_{v=1}^{m'} \sum_{p'=0}^{\infty} \theta(S_{p',v}) Z'^{S_{p,v}} x^{-\rho-\sigma-\rho'-\sigma'-k-(\eta'+\lambda')S_{p',v}} y^{-m-n-m'-n'-l-(\delta'+\mu')S_{p',v}} \left(\sigma' + \lambda' S_{p',v}\right)$$

$$\Gamma\left(\eta' + \mu' S_{p',v}\right) \int_0^x \int_0^y (x-u)^{\sigma+\sigma'+k+\lambda' S_{p',v}-1} (y-v)^{\eta+\eta'+l+\mu' S_{p',v}-1} u^{\rho'+\eta' S_{p',v}} v^{m'+\delta' S_{p',v}}$$

$$I_{p_i+4,q_i+4:r}^{m+2,n+2} \left[z \left(1-\frac{u}{x}\right)^{\lambda} \left(1-\frac{v}{y}\right)^{\mu} \middle| \begin{matrix} A^{**} \\ B^{**} \end{matrix} \right] f(u,v) du dv, \quad (3.1)$$

Where

$$A^{**} = (a_j, \alpha_j)_{1,n} (1-\sigma-k, \lambda), (1-\eta-l, \mu), (a_{ji}, \alpha_{ji})_{n+1, p_i},$$
$$(\rho'+\sigma'+(\eta'+\lambda')S_{p',v}, \eta), (m'+n'+(\delta'+\mu')S_{p',v}, \delta)$$
$$B^{**} = (b_j, \beta_j)_{1,m}, (\rho'+\sigma'+(\eta'+\lambda')S_{p',v}+k, \eta), (m'+n'+(\delta'+\mu')S_{p',v}+l, \delta)$$
$$(b_{ji}, \beta_{ji})_{m+1, q_i}, (1-\sigma-\sigma'-k-\lambda' S_{p',v}, \lambda), (1-\eta-\eta'-l-\mu S_{p',v}, \mu) \quad ,$$

And the following conditions are satisfied:

$$\min_{1\le k\le m} \mathrm{Re}\left[1+\rho+m+(\eta+\delta)\left(\frac{b_k}{\beta_k}\right)\right] > 0,\ \min_{1\le k\le m} \mathrm{Re}\left[\sigma+\eta+(\lambda+\mu)\left(\frac{b_k}{\beta_k}\right)\right] > 0,$$
$$\min_{1\le k\le m} \mathrm{Re}\left[1+\rho'+m'+(\eta'+\delta')\left(\frac{b'_k}{\beta'_k}\right)\right] > 0, \min_{1\le k\le m} \mathrm{Re}\left[\sigma'+\eta'+(\lambda'+\mu')\left(\frac{b'_k}{\beta'_k}\right)\right] > 0.$$

Similarly two dimensional formulae cab be obtained easily from result 1 and result 3. These formulae can be further reduced to results given by Raina [7,pp. 511-513, eqs. (2.8), (2.9) and (2.15)] by taking I -function to unity.
If in these composition formulas we reduce both the generalized class of polynomials, functions to unity, we arrive at the multidimensional analogue of the results given by Erdelyi [1, p. 166, eq. (6.2); p. 167, eq. (6.3)]. Again reducing the generalized hypergeometric function, we arrive at the corresponding result given by Goyal and Jain [2, p. 253, eq. (2.4); p. 254, eq. (2.7); p. 255, eq. (2.12)] after a little simplification. Further, if we reduce generalized class of polynomials to polynomials we arrive at the result which are in essence the same as those obtained by Goyal, Jain and Gaur [3,pp. 404-405, eq. (2.1); p. 406, eq. (2.7); pp.407-408, eq. (2.12)].

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