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Further Investigations On The Question Raised By Singh

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Abstract
In this paper we study the growth properties of composite entire and meromorphic functions which improve some earlier results.
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1. Introduction, Definitions and Notations.

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [3] proved that

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}=\infty \quad and \quad \limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)}=\infty.$$

Singh [13] proved some comparative growth properties of $\log T(r,f\circ g)$ and $T(r,f)$. He also raised the problem of investigating the comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$ which he was unable to solve. However, some results on the comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$ are proved in [9].

In the paper we further investigate the question of Singh [13] and prove some new results relating to the growth rates of composite entire and meromorphic functions improving some earlier results.

We do not explain the standard notations and definitions on the theory of entire and meromorphic functions because those are available in [14] and [8].

To start our paper we just recall the following definitions.

Definition 1 The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r} \quad and \quad \lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

If f is an entire function, one can easily verify that

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} \quad and \quad \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}.$$

If $\rho_f<\infty$ then f is of finite order. Also $\rho_f=0$ means that f is of order zero. In this connection Liao and Yang [11] gave the following definition.

Definition 2 [11] Let f be a meromorphic function of order zero. Then the quantities ρ_f^* and λ_f^* of a meromorphic function f are defined as:

$$\rho_f^*=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log^{[2]} r} \quad and \quad \lambda_f^*=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log^{[2]} r}.$$

If f is entire, then

$$\rho_f^*=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log^{[2]} r} \quad and \quad \lambda_f^*=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log^{[2]} r}.$$

Datta and Biswas [6] gave an alternative definition of zero order and zero lower order of a meromorphic function in the following way:

Definition 3 [6] Let f be a meromorphic function of order zero. Then the quantities ρ_f^{**} and λ_f^{**} of f are defined by:

$$\rho_f^{**}=\limsup_{r\to\infty}\frac{T(r,f)}{\log r} \quad and \quad \lambda_f^{**}=\liminf_{r\to\infty}\frac{T(r,f)}{\log r}.$$

If f is an entire function then clearly

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\log r} \quad and \quad \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r}.$$

Definition 4 The type σ_f and lower type $\overline{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} \quad and \quad \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When f is entire, it can be easily verified that

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}} \quad and \quad \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Datta and Jha [5] gave an definition of weak type of a meromorphic function of finite positive lower order in the following way:

Definition 5 [5] The weak type τ_f of a meromorphic function f of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}.$$

For entire f,

$$\tau_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$

Similarly one can define the growth indicator $\overline{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}.$$

When f is entire, it can be easily verified that

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] Let f be meromorphic and g be entire then for all sufficiently large values of r,

$$T(r, f\circ g) \leq \{1 + \circ(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g),f).$$

Lemma 2 [2] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) \geq T(exp(r^{\mu}), f).$$

Lemma 3 [10] Let f be meromorphic and g be entire such that $0 < \mu < \rho_g \leq \infty$ and $\lambda_f > 0$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) \geq T(exp(r^{\mu}), g).$$

Lemma 4 [7] Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) < T(exp(r^{\mu}), f).$$

Lemma 5 [7] Let f be a meromorphic function of finite order and g be an entire function with $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) < T(exp(r^{\mu}), g).$$

Lemma 6 [6] If f be any meromorphic function of order zero. Then (i) $\rho_f^* = 1$ and (ii) $\lambda_f^* = 1$.

Lemma 7 [4] Let f be meromorphic and g be transcendental entire such that $\rho_{f\circ g}$ and ρ_g are both finite. Then $\rho_{f\circ g} \leq \rho_f^* \rho_g$.

Lemma 8 [12] Let f be entire and g be a transcendental entire function of finite lower order. Then for any $\delta > 0$,

$$M\left(r^{1+\delta}, f \circ g\right) \geq M(M(r,g), f) \ \ (r \geq r_0).$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 1 Let f be a meromorphic function and g be an entire function such that $0 < \lambda_f \leq \rho_f < \rho_g < \infty$. Then

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r,f)} = \infty .$$

Proof. Since $\rho_f < \rho_g$ we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_f + \varepsilon < \rho_g - \varepsilon < \rho_g. \quad (1)$$

Now in view of Lemma 2 we obtain for a sequence of values of r tending to infinity that

$$\log T(r, f \circ g) \geq \log T(exp r^{(\rho_g - \varepsilon)}, f)$$

$$\text{i.e., } \log T(r, f \circ g) \geq (\lambda_f - \varepsilon) \log exp r^{(\rho_g - \varepsilon)}$$

$$\text{i.e., } \log T(r, f \circ g) \geq (\lambda_f - \varepsilon) r^{(\rho_g - \varepsilon)}. \quad (2)$$

Again from the definition of order, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log T(r,f) \leq (\rho_f + \varepsilon) \log r$$

$$\text{i.e., } T(r,f) \leq r^{(\rho_f + \varepsilon)}. \quad (3)$$

Therefore from (2) and (3) it follows for a sequence of values of r tending to infinity,

$$\frac{\log T(r, f \circ g)}{T(r,f)} \geq \frac{(\lambda_f - \varepsilon) r^{(\rho_g - \varepsilon)}}{r^{(\rho_f + \varepsilon)}}. \quad (4)$$

Now in view of (1) it follows from (4) that

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r,f)} = \infty .$$

This proves the theorem.

In the line of Theorem 1 the following corollary may be deduced:

Corollary 1 Let f be a meromorphic function and g be an entire function with $0 < \lambda_f \leq \rho_f < \lambda_g < \infty$.. Then

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r,f)} = \infty .$$

Remark 1 The condition $\lambda_f > 0$ in Theorem 1 and Corollary 1 is necessary which is evident from the following example.

Example 1 Let $f = z^2$ and $g = exp\, z$.
Then $\lambda_f = \rho_f = 0$ and $\rho_g = 1$, $f \circ g = exp 2z$.
Hence

$$\log T(r, f \circ g) \leq \log^{[2]} M(r, f \circ g) = \log r + O(1)$$

and

$$3T(2r,f) \geq \log M(r,f) = 2r$$

$$\text{i.e., } T(2r,f) \geq \frac{2}{3} r + O(1)$$

Therefore

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\le 1\,.$$

Theorem 2 Let f be a meromorphic function and g be an entire function such that (i) $0<\rho_f<\infty$, (ii) $\rho_f=\rho_g$, (iii) $\sigma_f>0$ and (iv) $\sigma_g<\infty$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\le\rho_f\frac{\sigma_g}{\sigma_f}=\rho_g\frac{\sigma_g}{\sigma_f}\,.$$

Proof. As $T(r,g)\le\log^{+}M(r,g)$, we have from Lemma 1 for all sufficiently large values of r,

$$T(r,f\circ g)\le\{1+o(1)\}T(M(r,g),f)$$

$$\text{i.e.,}\ \log T(r,f\circ g)\le(\rho_f+\varepsilon)\log M(r,g)+O(1)$$

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\le(\rho_f+\varepsilon)\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,f)}. \tag{5}$$

Also for all sufficiently large values of r

$$\log M(r,g)\le(\sigma_g+\varepsilon)r^{\rho_g}. \tag{6}$$

Again we obtain for a sequence of values of r tending to infinity that

$$T(r,f)\ge(\sigma_f-\varepsilon)r^{\rho_f}. \tag{7}$$

As $\rho_f=\rho_g$ we get from (6) and (7) that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,f)}\le\frac{\sigma_g}{\sigma_f} \tag{8}$$

Since $\varepsilon(>0)$ is arbitrary and $\rho_f=\rho_g$, from (5) and (8) we obtain that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\le\rho_f\frac{\sigma_g}{\sigma_f}=\rho_g\frac{\sigma_g}{\sigma_f}\,.$$

This completes the proof.

In the line of Theorem 2 the following corollary can also be proved.

Corollary 2 Let f be a meromorphic function and g be an entire function such that (i) $0<\lambda_f\le\rho_f<\infty$, (ii) $\rho_f=\rho_g$, (iii) $\bar{\sigma}_f>0$ and (iv) $\sigma_g<\infty$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\le min\left\{\rho_f\frac{\bar{\sigma}_g}{\bar{\sigma}_f},\lambda_f\frac{\sigma_g}{\bar{\sigma}_f}\right\}.$$

The following theorem can be proved in the line of Theorem 2 and so the proof is omitted.

Theorem 3 Let f be a meromorphic function and g be an entire function such that (i) $0<\rho_f<\infty$, (ii) $\rho_f=\rho_g$, (iii) $\bar{\sigma}_f>0$ and (iv) $\sigma_g<\infty$. Then

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\le\rho_f\frac{\sigma_g}{\bar{\sigma}_f}=\rho_g\frac{\sigma_g}{\bar{\sigma}_f}\,.$$

Theorem 4 Let f be a meromorphic function and g be an entire function such that $0<\lambda_g<\rho_f$, $0<\lambda_f\le\rho_f<\infty$, $\rho_g<\infty$ and $\bar{\sigma}_f>0$ .Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \le \min\left\{\frac{\rho_f}{\bar{\sigma}_f},\frac{\rho_g}{\bar{\sigma}_f}\right\}.$$

Proof. Since $\lambda_g < \rho_f$, in view of Lemma 4 we obtain for a sequence of values of r tending to infinity that

$$\log T(r,f\circ g) < \log T(exp(r^{\rho_f}),f)$$

$$\text{i.e.,}\ \log T(r,f\circ g) < (\rho_f+\varepsilon)\log exp(r^{\rho_f})$$

$$\text{i.e.,}\ \log T(r,f\circ g) < (\rho_f+\varepsilon)\, r^{\rho_f}. \tag{9}$$

Again we have for arbitrary positive ε and for all sufficiently large values of r,

$$T(r,f) \ge (\bar{\sigma}_f-\varepsilon) r^{\rho_f}. \tag{10}$$

Therefore from (9) and (10) it follows for a sequence of values of r tending to infinity

$$\frac{\log T(r,f\circ g)}{T(r,f)} \le \frac{(\rho_f+\varepsilon)\, r^{\rho_f}}{(\bar{\sigma}_f-\varepsilon) r^{\rho_f}}$$

$$\text{i.e.,}\quad \liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \le \frac{\rho_f}{\bar{\sigma}_f}. \tag{11}$$

Similarly in view of Lemma 5 we have for a sequence of values of r tending to infinity

$$\log T(r,f\circ g) < (\rho_g+\varepsilon)\, r^{\rho_f}. \tag{12}$$

So combining (10) and (12) we get for a sequence of values of r tending to infinity

$$\frac{\log T(r,f\circ g)}{T(r,f)} \le \frac{(\rho_g+\varepsilon)\, r^{\rho_f}}{(\bar{\sigma}_f-\varepsilon) r^{\rho_f}}$$

$$\text{i.e.,}\quad \liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \le \frac{\rho_g}{\bar{\sigma}_f}. \tag{13}$$

Thus the theorem follows from (11) and (13).

Using the notion of weak type, we may state the following three theorems without proof.

Theorem 5 Let f be a meromorphic function and g be an entire function such that (i) $0 < \lambda_f \le \rho_f < \infty$, (ii) $\lambda_f = \lambda_g$, (iii) $\tau_f > 0$ and (iv) $\bar{\tau}_g < \infty$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \le \min\left\{\rho_f\frac{\tau_g}{\tau_f},\lambda_f\frac{\bar{\tau}_g}{\tau_f},\rho_f\frac{\bar{\tau}_g}{\bar{\tau}_f}\right\}.$$

Theorem 6 Let f be a meromorphic function and g be an entire function such that $0 < \lambda_g < \lambda_f \le \rho_f < \infty$, $\rho_g < \infty$ and $\tau_f > 0$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \le \min\left\{\frac{\rho_f}{\tau_f},\frac{\rho_g}{\tau_f}\right\}.$$

Theorem 7 Let f be a meromorphic function and g be an entire function such that (i) $0 < \lambda_f \le \rho_f < \infty$, (ii) $\lambda_f = \lambda_g$, (iii) $\tau_f > 0$ and (iv) $\bar{\tau}_g < \infty$. Then

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \le \rho_f\frac{\bar{\tau}_g}{\tau_f}.$$

Theorem 8 Let f be a meromorphic function and g be an entire function such that $0 < \lambda_f < \rho_g < \infty$, $\lambda_g > 0$ and $\bar{\tau}_f < \infty$. Then

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\geq max\left\{\frac{\lambda_f}{\bar{\tau}_f},\frac{\lambda_g}{\bar{\tau}_f}\right\}.$$

Proof. Suppose $0 < \varepsilon < min\{\lambda_f,\lambda_g\}$.
Since $\lambda_f < \rho_g$, in view of Lemma 2 we obtain for a sequence of values of r tending to infinity that

$$\log T(r,f\circ g)\geq \log T(exp(r^{\lambda_f}),f)$$

$$\text{i.e.,}\ \log T(r,f\circ g)\geq (\lambda_f-\varepsilon)\log exp(r^{\lambda_f})$$

$$\text{i.e.,}\ \log T(r,f\circ g)\geq (\lambda_f-\varepsilon)r^{\lambda_f}. \tag{14}$$

Again we have for arbitrary positive ε and for all sufficiently large values of r,

$$\mathrm{T}(r,f)\geq(\bar{\tau}_f+\varepsilon)r^{\lambda_f}. \tag{15}$$

Therefore from (14) and (15) it follows for a sequence of values of r tending to infinity

$$\frac{\log T(r,f\circ g)}{T(r,f)}\geq\frac{(\lambda_f-\varepsilon)r^{\lambda_f}}{(\bar{\tau}_f+\varepsilon)r^{\lambda_f}}$$

$$\text{i.e.,}\quad \limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\geq\frac{\lambda_f}{\bar{\tau}_f}. \tag{16}$$

Similarly in view of Lemma 3 we have for a sequence of values of r tending to infinity

$$\log T(r,f\circ g)\geq(\lambda_g-\varepsilon)r^{\lambda_f}. \tag{17}$$

So combining (15) and (17) we get for a sequence of values of r tending to infinity

$$\frac{\log T(r,f\circ g)}{T(r,f)}\geq\frac{(\lambda_g-\varepsilon)r^{\lambda_f}}{(\bar{\tau}_f+\varepsilon)r^{\lambda_f}}$$

Since ε (> 0) is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)}\geq\frac{\lambda_g}{\bar{\tau}_f}. \tag{18}$$

Thus the theorem follows from (16) and (18).

We may now state the following four corollaries without proof for the right factor g of the composite function $f\circ g$ based on type and weak type of an entire function.

Corollary 3 Let f be a meromorphic function and g be an entire function such that $0<\lambda_g<\lambda_f\leq\rho_f<\infty$, $\rho_g<\infty$ and $\tau_g>0$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)}\leq min\left\{\frac{\rho_f}{\tau_g},\frac{\rho_g}{\tau_g}\right\}.$$

Corollary 4 Let f be a meromorphic function and g be an entire function such that $0<\lambda_g<\rho_g<\infty$, $\lambda_f>0$ and $\bar{\tau}_g<\infty$. Then

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)}\geq max\left\{\frac{\lambda_f}{\bar{\tau}_g},\frac{\lambda_g}{\bar{\tau}_g}\right\}.$$

Corollary 3 and Corollary 4 can easily be proved in the line of Theorem 6 and Theorem 8 respectively. So the proof is omitted.

Remark 2 Under the same conditions of Corollary 4 if f be a meromorphic function with lower order zero, then with the help of λ_f^{**} (> 0) and similar process of Theorem 8 one can easily verify that

$$\limsup_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)} \geq \frac{\lambda_f^{**}}{\bar{\tau}_g}.$$

Corollary 5 Let f be a meromorphic function and g be an entire function such that (i) $0 < \rho_f < \infty$, (ii) $0 < \lambda_g \leq \rho_g < \infty$, (iii) $0 < \bar{\sigma}_g \leq \sigma_g < \infty$ and (iv) $0 < \bar{\tau}_g \leq \tau_g < \infty$. Then

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \leq min\left\{\rho_f\frac{\sigma_g}{\bar{\sigma}_g}, \rho_f\frac{\bar{\tau}_g}{\tau_g}\right\}.$$

The proof is omitted as it can be carried out in the line of Theorem 3and Theorem 7.

Remark 3 In addition to the conditions of Corollary 5 if f be a meromorphic function with order zero and $0 < \rho_f^{**} < \infty$ then by Definition 3 and similar process of Theorem 3 and Theorem 7, one can verify that

$$\limsup_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)} \leq \{1+o(1)\}\rho_f^{**}.min\left\{\frac{\sigma_g}{\bar{\sigma}_g}, \frac{\bar{\tau}_g}{\tau_g}\right\}.$$

Remark 4 If we take $0 < \lambda_f < \infty$ instead of $0 < \rho_f < \infty$ in Corollary 5 and the other conditions remain the same then it can be shown in the line of Corollary 2 and Theorem 5 that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,f)} \leq min\left\{\lambda_f\frac{\sigma_g}{\bar{\sigma}_g}, \lambda_f\frac{\bar{\tau}_g}{\tau_g}\right\}.$$

Remark 5 In Remark 3, if we take $0 < \lambda_f^{**} < \infty$ instead of $0 < \rho_f^{**} < \infty$ and the other conditions remain the same then it can be shown that

$$\liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)} \leq \{1+o(1)\}\lambda_f^{**} min\left\{\frac{\sigma_g}{\bar{\sigma}_g}, \frac{\bar{\tau}_g}{\tau_g}\right\}.$$

Corollary 6 Let f be a meromorphic function and g be an entire function such that that $0 < \lambda_f \leq \rho_f < \infty$, $0 < \lambda_g < \rho_f$ and $\bar{\sigma}_g > 0$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)} \leq min\left\{\frac{\rho_f}{\bar{\sigma}_g}, \frac{\rho_g}{\bar{\sigma}_g}\right\}.$$

Corollary 6 can easily be proved in the line of Theorem 4. Hence the proof is omitted.

Theorem 9 Let f be a meromorphic function of order zero and g be an entire function of non zero order. Then

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(exp\,(r^A),f)} \geq \frac{\rho_g}{A}$$

where $A > 0$.

Proof. In view of Lemma 2 we obtain for a sequence of values of r tending to infinity that

$$\log T(r,f\circ g) \geq \log T(exp r^{(\rho_g-\varepsilon)},f)$$

$$\text{i.e.,}\ \log T(r,f\circ g) \geq (\lambda_f^* - \varepsilon)\log^{[2]} exp r^{(\rho_g-\varepsilon)}$$

$$\text{i.e.,}\ \log T(r,f\circ g) \geq (\lambda_f^* - \varepsilon)(\rho_g-\varepsilon)\log r. \tag{19}$$

Again from the definition of ρ_f^*, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log T(exp(r^A), f) \leq (\rho_f^* + \varepsilon) \log^{[2]} exp(r^A)$$

$$\text{i.e.,} \quad \log T(exp(r^A), f) \leq A(\rho_f^* + \varepsilon) \log r. \tag{20}$$

Therefore from (19) and (20) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T(r, f \circ g)}{\log T(exp(r^A), f)} \geq \frac{(\lambda_f^* - \varepsilon)(\rho_g - \varepsilon) \log r}{A(\rho_f^* + \varepsilon) \log r}. \tag{21}$$

Since $\varepsilon\ (>0)$ is arbitrary, by Lemma 6 the theorem follows from (21) .

Remark 6 If we take $0 < A < \rho_g$ instead of "$A > 0$" in Theorem 9 and the other conditions remain the same then with the help of ρ_f^{**} and λ_f^{**} the following holds:

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(exp(r^A), f)} = \infty.$$

In the line of Theorem 9 the following corollary may be deduced:

Corollary 7 Let f be a meromorphic function of order zero and g be an entire function of non zero finite order. Then

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, g)} \geq 1.$$

Now we present the next two theorems which will be needed in the sequel.

Theorem 10 Let f be meromorphic and g be entire such that $\rho_f = 0$ .. Then $\rho_{f \circ g} \geq \rho_g$.

Proof. Let us choose $\varepsilon\ (>0)$ such that $\mu = (\rho_g - \varepsilon)$.

Now from (19) we get for a sequence of values of r tending to infinity that

$$\frac{\log T(r, f \circ g)}{\log r} \geq (\lambda_f^* - \varepsilon)(\rho_g - \varepsilon).$$

Since $\varepsilon\ (>0)$ is arbitrary, it follows from Lemma 6 and above that

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log r} \geq \lambda_f^* \rho_g = 1 \rho_g = \rho_g$$

$$\text{i.e.,} \quad \rho_{f \circ g} \geq \rho_g.$$

Thus the theorem is established.

Theorem 11 Let f be an entire function with $\lambda_f = 0$ and g be a transcendental entire function of finite lower order. Then $\lambda_{f \circ g} \geq \lambda_g$.

Proof. From Lemma 6 and Lemma 8 it follows that

$$\lambda_{f \circ g} = \liminf_{r \to \infty} \frac{M(r^{1+\delta}, f \circ g)}{\log r^{1+\delta}}$$

$$\geq \liminf_{r \to \infty} \frac{M(M(r, g), f)}{\log^{[2]} M(r, g)} . \liminf_{r \to \infty} \frac{\log^{[2]} M(r, g)}{\log r}$$

$$= \lambda_f^* \lambda_g = 1 \lambda_g = \lambda_g.$$

This proves the theorem.

Theorem 12 Let f, h be two meromorphic functions and g, k be any two entire functions such that $\rho_f = 0$, $\rho_g < \infty$ and $\lambda_{h \circ k}$ is positive. Then for every positive constant A

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, (h \circ k)^l)} \leq \frac{\rho_g}{A \ \lambda_{h \circ k}},$$

where $l = 0, 1, 2, \ldots$.

Proof. Since $T(r,g) \leq \log^{+} M(r,g)$, we have from Lemma 1 for all sufficiently large values of r,

$$T(r, f\circ g) \leq \{1 + o(1)\} T(M(r,g), f)$$

$$\text{i.e., } T(r, f\circ g) \leq \{1 + o(1)\}(\rho_f^{**} + \varepsilon)\log M(r,g)$$

$$\text{i.e., } \log T(r, f\circ g) \leq \log^{[2]} M(r,g) + O(1)$$

$$\text{i.e., } \log T(r, f\circ g) \leq \left(\rho_g + \varepsilon\right)\log r + O(1). \tag{22}$$

Again for all sufficiently large values of r,

$$\log T\left(r^A, (h\circ k)^l\right) \geq (\lambda_{h\circ k} - \varepsilon)\log r^A$$

$$\text{i.e., } \log T\left(r^A, (h\circ k)^l\right) \geq A(\lambda_{h\circ k} - \varepsilon)\log r. \tag{23}$$

Therefore from (22) and (23) we have for all sufficiently large values of r that

$$\frac{\log T(r, f\circ g)}{\log T\left(r^A, (h\circ k)^l\right)} \leq \frac{\left(\rho_g + \varepsilon\right)\log r + O(1)}{A(\lambda_{h\circ k} - \varepsilon)\log r}$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T\left(r^A, (h\circ k)^l\right)} \leq \frac{\rho_g}{A\,\lambda_{h\circ k}}.$$

Thus the theorem is established.

Remark 7 If we take " $\rho_{h\circ k}$ is positive" instead of " $\lambda_{h\circ k}$ is positive" in Theorem 12 and the other conditions remain the same then it can be carried out that

$$\limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r^A, (h\circ k)^l)} \leq \frac{\rho_g}{A\,\rho_{h\circ k}}.$$

Remark 8 Under the same conditions of Theorem 12 if h be an entire function with $\lambda_h = 0$ and k be a transcendental entire function of finite lower order, then with the help of Theorem 11 and in the line of Theorem 12 one can easily verify that

$$\limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r^A, (h\circ k)^l)} \leq \frac{\rho_g}{A\,\lambda_k}.$$

Remark 9 Under the same conditions of Remark 7 if h be entire function with $\rho_h = 0$, then with the help of Theorem 10 and in the line of Theorem 12 one can easily verify that

$$\limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r^A, (h\circ k)^l)} \leq \frac{\rho_g}{A\,\rho_k}.$$

Theorem 13 Let f, h be meromorphic functions and g, k be entire functions such that $\lambda_f = 0$, $\rho_g > 0$ and $0 < \rho_{h\circ k} < \infty$. Then for every positive constant A

$$\limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r^A, (h\circ k)^l)} \geq \frac{\rho_g}{A\,\rho_{h\circ k}}$$

where $l = 0, 1, 2, \ldots$

Proof. Let us choose ε (> 0) such that $0<\mu<\min\{1, \rho_g\}$ and $\mu = (\rho_g - \varepsilon)$.

Now for all sufficiently large values of r,

$$\log T\left(r^A, (h\circ k)^l\right) \leq \left(\rho_{h\circ k} + \varepsilon\right)\log r^A$$

$$\text{i.e., } \log T\left(r^A, (h\circ k)^l\right) \leq A\left(\rho_{h\circ k} + \varepsilon\right)\log r. \tag{24}$$

Therefore from (19) and (24) we have for a sequence of values of r tending to infinity that

$$\frac{\log T(r, f\circ g)}{\log T\left(r^{A}, (h\circ k)^{l}\right)} \geq \frac{(\lambda_f^* - \varepsilon)(\rho_g - \varepsilon)\log r}{A(\rho_{h\circ k} + \varepsilon)\log r}.$$

Since ε (> 0) is arbitrary, by Lemma 6 the theorem follows from above.

Remark 10 In addition to the conditions of Theorem 13, if k be entire with $\rho_k < \infty$ and $\rho_h = 0$then with the help of Lemma 6 and Lemma 7 and in the line of Theorem 13, one can easily verify that

$$\limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r^{A}, (h\circ k)^{l})} \geq \frac{\rho_g}{A\,\rho_k}.$$

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