EXISTENCE OF NONOSCILLATORY SOLUTIONS OF SECOND ORDER LINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH FORCING TERM

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Abstract. This paper is concerned with the nonoscillation of second order linear neutral delay difference equations with forcing term. By using Banach's contraction mapping principle, authors obtain some sufficient conditions for the existence of nonoscillatory solutions. To support the results, we use MATLAB programming to illustrate the examples.

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1. INTRODUCTION

In this paper, we consider the second order neutral delay difference equation with forcing term of the form

$$\Delta\left(r(n)\Delta[x(n)+p(n)x(n-\tau)]\right)+q_1(n)x(n-\sigma_1)-q_2(n)x(n-\sigma_2)=e(n)\ , \quad (1)$$

where $n\geq n_0$, $\tau>0$, $\sigma_1,\sigma_2\geq 0$ are integers. We assume the following conditions:

$(A1)$ $q_i>0$ and $\sum_{s=n_0}^{\infty} sq_i(n)<\infty, i=1,2$

$(A2)$ There exists a function $E(n)\in C^2([n_0,\infty),R)$ such that $\Delta^2(E(n))=e(n)$ and $\lim_{n\to\infty} E(n)=M\in R.$

The nonoscillatory behavior of linear and nonlinear neutral delay difference and differential equations with positive and negative coefficients have been investigated by several authors, see, for example [3], [4], [5], [6], [7], [8] and [9], the references cited therein. We refer monographs [1] and [2] for good amount of discussion concerning the existence of solution of delay difference equations. Our aim in this paper is to establish the nonoscillation criteria for the second order linear neutral delay difference equation (1) for various ranges of $p(n)$

As is customary, a solution of equation (1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory. We define $\rho=\max\{\tau,\sigma_1,\sigma_2\}$

Theorem1.1. (Banach's Contraction Principle [2]). Let (X ,d) be a complete metric space and let T be a contraction mapping on X. Then T has exactly one fixed point on X, that is, there exists exactly one $x\in X$ such that $Tx=x$

2. Main Results

Theorem 2.2. Suppose that conditions (A_1) and (A_2) hold and that there exists a constant p_1 such that

$$0 < p(n) \le p_1 < 1. \tag{2}$$

Then equation (1) has a nonoscillatory solution.

Proof. Suppose (2) holds. Choose constants $N_1 \ge M_1 > 0$ and choose $n_1 > n_0 + \rho$, sufficiently large such that

$$\sum_{s=n_1}^{\infty} s\left[q_1(s) + q_2(s)\right] < \frac{3(1-p_1)}{4}, \tag{3}$$

$$\sum_{s=n_1}^{\infty} s\, q_1(s) \le \frac{p_1 + N_1 - 1}{N_1}, \tag{4}$$

$$\sum_{s=n_1}^{\infty} s\, q_2(s) \le \frac{1 - p_1(1+2N_1) - 2M_1}{2N_1} \text{ and} \tag{5}$$

$$\left|E(n) - M\right| \le \frac{1-p_1}{4}. \tag{6}$$

Let X be the set of all continuous and bounded functions on $[n_0, \infty)$ with the sup norm. Define

$$A_1 = \left\{ x \in X : M_1 \le x(n) \le N_1, n \ge n_0 \right\}.$$

Define a mapping $T_1 : A_1 \to X$ as follows:

$$(T_1x)(n) = \begin{cases} \dfrac{3(1-p_1)}{4} - p(n)x(n-\tau) \\ +(n-1)\sum_{s=n}^{\infty}[q_1(s)x(s-\sigma_1) - q_2(s)x(s-\sigma_2)] \\ +\sum_{s=n_1}^{n-1} s[q_1(s)x(s-\sigma_1) - q_2(s)x(s-\sigma_2)] + E(n) - M,\ n \ge n_1 \\ (T_1x)(n_1),\ n_0 \le n \le n_1. \end{cases}$$

Clearly, T_1x is continuous. For every $x \in A_1$ and $n \ge n_1$ using (4) and (6), we obtain

$$(T_1x)(n) \le 1 - p_1 + N_1 \sum_{s=n_1}^{\infty} sq_1(s) \le N_1.$$

Furthermore, in view of (5) and (6), we have

$$(T_1x)(n) \ge \frac{1-p_1}{2} - p_1N_1 - N_1 \sum_{s=n_1}^{\infty} sq_2(s) \ge M_1.$$

Thus we proved that $T_1A_1 \subset A_1$. Since A_1 is a bounded, closed and convex subset of X , we have to prove that T_1 is a contraction mapping on A_1 to apply the contraction principle.

Now, for $x_1, x_2 \in A_1$ and $n \ge n_1$, in view of (3), we have

$$\begin{aligned}\left|(T_1x_1)(n)-(T_1x_2)(n)\right| &\le \|x_1-x_2\|\left\{p_1+\sum_{s=n_1}^{\infty} s\left[q_1(s)+q_2(s)\right]\right\}\\ &< \frac{3+p_1}{4}\|x_1-x_2\|\\ &= \lambda_1\|x_1-x_2\|.\end{aligned}$$

This implies that $\|T_1x_1-T_1x_2\| < \lambda_1\|x_1-x_2\|,\ \lambda_1<1.$ This proves that T_1 is a contraction mapping on A_1. T_1 has the unique fixed point $x,$ which is obviously a positive solution equation (1). This completes the proof.

Theorem 2.2. Suppose that conditions (A_1) and (A_2) hold and that there exists a constant p_2 such that

$$-1<-p_2 \le p(n)<0. \tag{7}$$

Then equation (1) has a nonoscillatory solution.

Proof. Suppose (7) holds. Choose constants $N_2 \ge M_2 > 0$ and choose $n_1 > n_0+\rho,$ sufficiently large such that

$$\sum_{s=n_1}^{\infty} s\left[q_1(s)+q_2(s)\right] < \frac{3(1-p_2)}{4}, \tag{8}$$

$$\sum_{s=n_1}^{\infty} s\,q_1(s) \le \frac{(1-p_2)(N_2-1)}{N_2}, \tag{9}$$

$$\sum_{s=n_1}^{\infty} s\,q_2(s) \le \frac{(1-p_2)-2M_2}{N_2}\ and \tag{10}$$

$$\left|E(n)-M\right| \le \frac{1-p_2}{4}. \tag{11}$$

Let X be the set as in theorem 1. Define

$$A_2=\left\{x\in X : M_2 \le x(n) \le N_2, n \ge n_0\right\}.$$

Define a mapping $T_2 : A_2 \to X$ as follows:

$$(T_2x)(n)=\begin{cases}\dfrac{3(1-p_2)}{4}-p(n)x(n-\tau)\\ +(n-1)\sum_{s=n}^{\infty}[q_1(s)x(s-\sigma_1)-q_2(s)x(s-\sigma_2)]\\ +\sum_{s=n_1}^{n-1}s[q_1(s)x(s-\sigma_1)-q_2(s)x(s-\sigma_2)]+E(n)-M,\ n\geq n_1\\ (T_2x)(n_1),\ n_0\leq n\leq n_1.\end{cases}$$

Clearly, T_2x is continuous. For every $x\in A_2$ and $n\geq n_1$ using (9) and (11), we obtain

$$\begin{aligned}(T_2x)(n)&\leq 1-p_2+p_2N_2+N_2\sum_{s=n_1}^{\infty}sq_1(s)\\ &\leq N_2, n\geq n_1.\end{aligned}$$

Furthermore, in view of (10) and (11), we have

$$\begin{aligned}(T_2x)(n)&\geq \frac{1-p_2}{2}-N_2\sum_{s=n_1}^{\infty}sq_2(s)\\ &\geq M_2,\ n\geq n_1.\end{aligned}$$

Thus we proved that $T_2A_2\subset A_2$. Since A_2 is a bounded, closed and convex subset of X , we have to prove that T_2 is a contraction mapping on A_2 to apply the contraction principle.

Now, for $x_1,x_2\in A_2$ and $n\geq n_1$, in view of (8), we have

$$\begin{aligned}\left|(T_2x_1)(n)-(T_2x_2)(n)\right|&\leq p_2\|x_1-x_2\|+\|x_1-x_2\|\sum_{s=n_1}^{\infty}s\left[q_1(s)+q_2(s)\right]\\ &<\frac{p_2+3}{4}\|x_1-x_2\|\\ &=\lambda_2\|x_1-x_2\|.\end{aligned}$$

This implies that $\|T_2x_1-T_2x_2\|<\lambda_2\|x_1-x_2\|,\ \lambda_2<1.$ This proves that T_2 is a contraction mapping on A_2. Consequently T_2 has the unique fixed point x, which is obviously a positive solution of equation (1). This completes the proof.

Theorem 2.3. *Suppose that conditions $(A1)$ and $(A2)$ hold and that there exist constants p_1 and p_2 such that*

$$1<p_2\leq p(n)<p_1<+\infty \tag{12}$$

Then equation (1) has a non oscillatory solution.

Proof. Suppose that (12) holds. Choose constants $N_3 \ge M_3 > 0$ and choose $n_1 > n_0 + \rho$ sufficiently large such that

$$\sum_{s=n_1}^{\infty} s[q_1(s)+q_2(s)] < \frac{3(p_2-1)}{4}, \tag{13}$$

$$\sum_{s=n_1}^{\infty} sq_1(s) \le \frac{1-p_2(1-N_3)}{N_3}, \tag{14}$$

$$\sum_{s=n_1}^{\infty} sq_2(s) \le \frac{p_2(p_2-1)-2p_1(N_3+p_2M_3)}{2p_1N_3} \quad and \tag{15}$$

$$|E(n)-M| \le \frac{p_2-1}{4}. \tag{16}$$

Let X be the set as in Theorem 1. Define

$$A_3 = \{x \in X : M_3 \le x(n) \le N_3, n \ge n_0\}.$$

Define a mapping $T_3 : A_3 \to X$ as follows:

$$(T_3x)(n) = \begin{cases} \dfrac{3(p_2-1)}{4p(n+\tau)} - \dfrac{x(n+\tau)}{p(n+\tau)} \\ +\dfrac{n-1+\tau}{p(n+\tau)} \sum_{s=n+\tau}^{\infty} [q_1(s)x(s-\sigma_1)-q_2(s)x(s-\sigma_2)] \\ +\dfrac{1}{p(n+\tau)} \sum_{s=n_1}^{n+\tau-1} s[q_1(s)x(s-\sigma_1)-q_2(s)x(s-\sigma_2)] + \dfrac{E(n+\tau)-M}{p(n+\tau)}, \; n \ge n_1 \\ (T_3x)(n_1), \; n_0 \le n \le n_1 \end{cases}$$

Clearly, T_3x is continuous. For every $x \in A_3$ and $n \ge n_1$ using (14) and (16), we get

$$(T_3x)(n) \le 1 - \frac{1}{p_2} + \frac{N_3}{p_2} \sum_{s=n_1}^{\infty} sq_1(s)$$

$$\le N_3, \quad n \ge n_1.$$

Furthermore in view of (15) and (16), we have

$$(T_3x)(n) \ge \frac{p_2-1}{2p_1} - \frac{N_3}{p_2} - \frac{N_3}{p_2} \sum_{s=n_1}^{\infty} sq_2(s)$$

$$\ge M_3, \quad n \ge n_1$$

Thus we proved that $T_3 A_3 \subset A_3$. Since A_3 is a bounded, closed and convex subset of X we have to prove that T_3 is a contraction mapping on A_3 to apply the contraction principle.

Now, for $x_1, x_2 \in A_3$ and $n \geq n_1$, where in view of (13), we obtain

$$\begin{aligned}
|(T_3x_1)(n)-(T_3x_2)(n)| &\leq \frac{1}{p_2}\|x_1-x_2\|\left\{1+\sum_{s=n_1}^{\infty}s[q_1(s)+q_2(s)]\right\} \\
&< \frac{3p_2+1}{4p_2}\|x_1-x_2\| \\
&= \lambda_3\|x_1-x_2\|.
\end{aligned}$$

This implies that $\|T_3x_1 - T_3x_2\| < \lambda_3 \|x_1 - x_2\|$, we have $\lambda_3 < 1$. This proves that T_3 is contraction mapping on A_3. Consequently T_3 has the unique fixed point $x \in X$, which is obviously a positive solution of equation (1) . This completes the proof.

Theorem 2.4. *Suppose that the conditions $(A1)$ and $(A2)$ hold and that there exist constants p_1 and p_2 such that*

$$-\infty < -p_2 < p(n) \leq -p_1 < -1. \tag{17}$$

Then equation (1) has a nonoscillatory solution.

Proof. Suppose (17) holds, Choose constants $N_4 \geq M_4 > 0$ and choose $n_1 > n_0 + \rho,$ sufficiently large such that

$$\sum_{s=n_1}^{\infty} s[q_1(s)+q_2(s)] < \frac{3(p_1-1)}{4}, \tag{18}$$

$$\sum_{s=n_1}^{\infty} sq_1(s) \leq \frac{p_2 - p_1(1+M_4)(p_2-1)}{p_2N_4}, \tag{19}$$

$$\sum_{s=n_1}^{\infty} sq_2(s) \leq \frac{N_4(p_1-1)-p_1}{N_4} \quad and \tag{20}$$

$$|E(n)-M| \leq p_1 - 1 \tag{21}$$

Let X be the set as in theorem 1. Define

$$A_4 = \{x \in X : M_4 \leq x(n) \leq N_4, n \geq n_0\}.$$

Define a mapping $T_4 : A_4 \to X$ as follows:

$$(T_4x)(n) = \begin{cases} -\dfrac{1}{p(n+\tau)} - \dfrac{x(n+\tau)}{p(n+\tau)} \\ +\dfrac{n-1+\tau}{p(n+\tau)} \sum_{s=n+\tau}^{\infty} [q_1(s)x(s-\sigma_1) - q_2(s)x(s-\sigma_2)] \\ +\dfrac{1}{p(n+\tau)} \sum_{s=n_1}^{n+\tau-1} s[q_1(s)x(s-\sigma_1) - q_2(s)x(s-\sigma_2)] + \dfrac{E(n+\tau)-M}{p(n+\tau)},\ n \ge n_1 \\ (T_4x)(n_1),\ n_0 \le n \le n_1. \end{cases}$$

Clearly, T_4x is continuous. For every $x \in A_4$ and $n \ge n_1$ using (20) and (21), we obtain

$$\begin{aligned}(T_4x)(n) &\le -\left(\frac{1+x(n+\tau)}{p(n+\tau)}\right) - \frac{1}{p(n+\tau)} \sum_{s=n_1}^{\infty} sq_2(s)x(s-\sigma_2) + \frac{E(n+\tau)-M}{p(n+\tau)} \\ &\le \frac{1+N_4}{p_1} + \frac{N_4}{p_1} \sum_{s=n_1}^{\infty} sq_2(s) + \frac{p_1-1}{p_1} \\ &\le N_{4,} \quad n \ge n_1.\end{aligned}$$

Furthermore in view of (19) and (20) we have

$$\begin{aligned}(T_4x)(n) &\ge -\left(\frac{1+x(n+\tau)}{p(n+\tau)}\right) + \frac{1}{p(n+\tau)} \sum_{s=n_1}^{\infty} sq_1(s)x(s-\sigma_1) + \frac{E(n+\tau)-M}{p(n+\tau)} \\ &\ge \frac{1+M_4}{p_2} - \frac{N_4}{p_1} \sum_{s=n_1}^{\infty} sq_1(s) - \frac{p_1-1}{p_1} \\ &\ge M_4.\ \ n \ge n_1\end{aligned}$$

Thus we proved that $T_4A_4 \subset A_4$. Since A_4 is a bounded, closed and convex subset of X, we have to prove that T_4 is a contraction mapping on A_4 to apply the contraction principle.

Now, for $x_1, x_2 \in A_4$ and $n \ge n_1$, in view of (18), we have

$$\begin{aligned}|(T_4x_1)(n) - (T_4x_2)(n)| &\le \frac{1}{p_1} \| x_1 - x_2 \| \left\{ 1 + \sum_{s=n_1}^{\infty} s[q_1(s) + q_2(s)] \right\} \\ &< \frac{3p_1+1}{4p_1} \| x_1 - x_2 \| \\ &= \lambda_4 \| x_1 - x_2 \|.\end{aligned}$$

This implies that $\| T_4x_1 - T_4x_2 \| < \lambda_4 \| x_1 - x_2 \|$, we get $\lambda_4 < 1$.

This proves that T_4 is contraction mapping on A_4. T_4 has the unique fixed point $x \in X$, which is obviously a positive solution of equation (1). This completes the proof.

Example 2.5. Consider the second order delay difference equation

$$\Delta^2(x(n)+e^{-n}x(n-1))+2e^{-n-2}x(n-1)-e^{-n}x(n-1)=e(n), \qquad (22)$$

where $p(n)=e^{-n}, q_1(n)=2e^{-n-2}, q_2(n)=e^{-n}, e(n)=e^{-n-2}-2e^{-n-1}+e^{-n}+e^{-2n-3}$ such that $0<p(n)\le e^{-n}<1$. Since $E(n)\to 0$ as $n\to\infty$, $\sum_{s=n_1}^{\infty} sq_1(n)<\infty$ and $\sum_{s=n_1}^{\infty} sq_2(n)<\infty$. Then the sufficient conditions of Theorem 2.1 are satisfied. Therefore, the equation (22) has a positive solution.

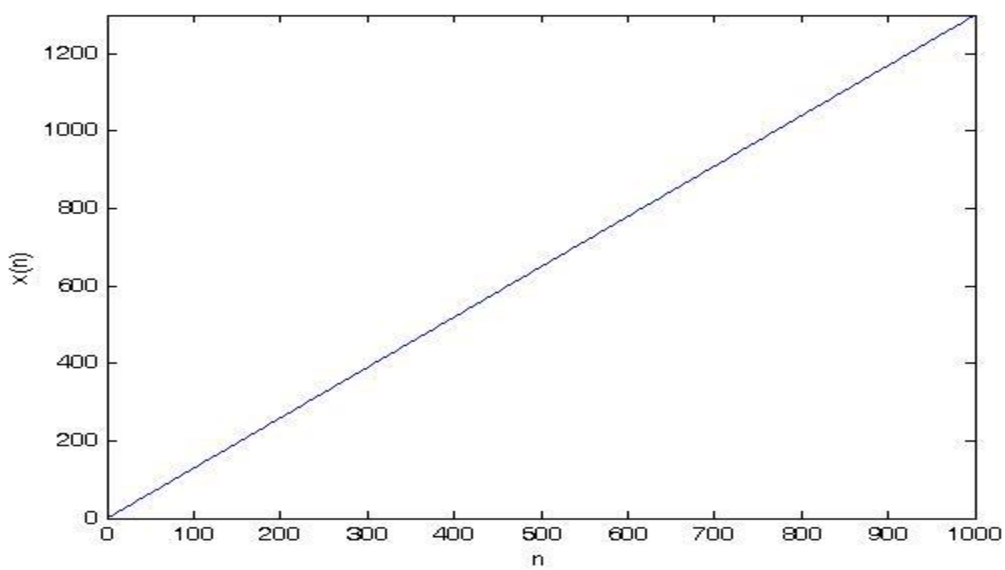

Figure 1

In Figure 1, the Nonoscillatory nature of equation (22) can be seen easily.

Example 2.6. Consider the second order delay difference equation.

$$\Delta^2(x(n)-(1+e^{-n})x(n-1))+e^{-n}x(n-1)-2e^{-n-2}x(n-1)=e(n), \qquad (23)$$

Where $p(n)=-1-e^{-n}, q_1(n)=e^{-n}, q_2(n)=2e^{-n-2}, e(n)=e^{-n-2}-3e^{-n-1}+3e^{-n}-e^{-n+1}+e^{-2n-3}$

such that $-\infty<p(n)\le -1$. Since $E(n)\to 0$ as $n\to\infty$, $\sum_{s=n_1}^{\infty} sq_1(n)<\infty$ and $\sum_{s=n_1}^{\infty} sq_2(n)<\infty$. Then the sufficient conditions of Theorem 2.4 are satisfied. Therefore, the equation (23) has a positive solution.

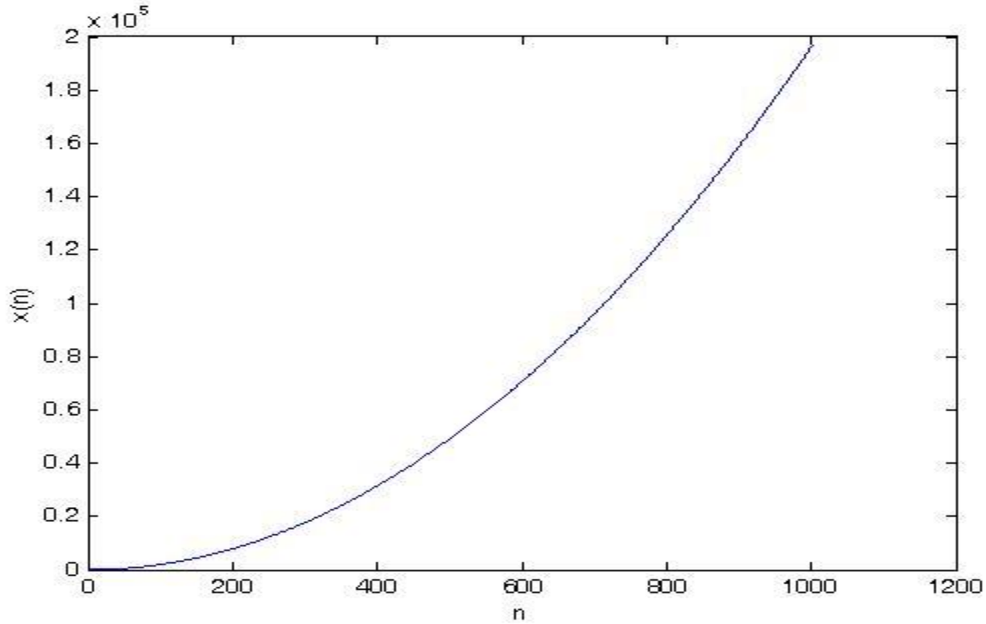

Figure 2

In Figure 2, the Nonoscillatory nature of equation (23) can be seen easily.

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