On Center of Automorphisms Group

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Abstract: Let G be a group and p be a prime integer. In this paper we prove a theorem on order of $Z(Aut(G))$ when $|Z(G)| = p$.

Key word: center of a group , automorphisms group.

I. INTRODUCTION

In this paper G is a group and p denotes prime number. The center of a group G, denoted Z(G) and automorphism group of G denoted by $Aut(G)$. We first give preliminary information and then we characterize order center of $Aut(G)$ when $|Z(G)| = p$.

II. PRELIMINARY RESULTS

Definition 2.1: Let G be a group. The set of elements that commute with every element of G called the center of G and denoted by Z(G) . On the other hand,

$$Z(G) = \{z \in G \mid zg = gz, \forall g \in G\}$$

For $H \le G$, we define the centrlizer of H in G to be $C_G(H) = \{g \in G \mid gh = hg, \forall h \in H\}$, then $Z(G) \le C_G(H) \le G$.

Definition 2.2: Suppose that $x, g \in G$ and write $x^g = g^{-1}xg$, this element is called the conjugate of x by g.

Definition 2.3. If G is a group, an automorphism of G is an isomorphism from G to G. The set of automorphisms of G is denoted by $Aut(G)$.

Theorem 2.4: : If H act on K. Then, to each $h \in H$ there corresponds a map $\varphi_h : K \to K$, defined by $\varphi_h : k \mapsto k^h$, and this is an automorphism of K.Moreover, the map $\varphi : H \to Aut(K)$, defined by $\varphi : h \mapsto \varphi_h$ is a homomorphism.

Proof: See [3-7] theorem 9.3.

III. MAIN THEOREM

Main Theorem : If G is a group with $|Z(G)| = p$, then $Z(Aut(G))^{p(p-1)} = \langle I \rangle$.

Proof: Let $\psi \in Z(Aut(G))$ then $\psi\psi_g = \psi_g\psi$ for all $g \in G$, where $\begin{matrix}\psi_g : G \to G \\ x \mapsto x^g\end{matrix}$. Now we have

$$\psi(\psi_g(x)) = \psi_g(\psi(x)) \Rightarrow \psi(x^g) = \psi(x)^g \Rightarrow \psi(x)^{\psi(g)} = \psi(x)^g \Rightarrow \psi(x)^{\psi(g)g^{-1}} = \psi(x) \Rightarrow [\psi(x), \psi(g)g^{-1}] = e$$

for all $x, g \in G$, but $\psi(G) = G$ so $[G, \psi(g)g^{-1}] = [\psi(G), \psi(g)g^{-1}] = e$ i.e $\psi(g)g^{-1} \in Z(G)$ for all $g \in G$.

Let $\psi \in Z(Aut(G))$ be fix and $\theta(g) = \psi(g)g^{-1} \in Z(G)$ for all $g \in G$, we have

$$\theta(xy) = \psi(xy)(xy)^{-1} = \psi(x)\psi(y)y^{-1}x^{-1} = \psi(x)\theta(y)x^{-1} = \psi(x)x^{-1}\theta(y) = \theta(x)\theta(y)$$

for all $x, y \in G$. Therefore $\begin{matrix}\theta : G \to G \\ g \mapsto \psi(g)g^{-1}\end{matrix}$ is a homomorphism.

By the above $\psi(g) = g\theta(g)$, $\forall g \in G$, now by induction we can prove that

$$\psi^{(n)}(g) = g^{\binom{n}{0}}\theta(g)^{\binom{n}{1}}\ldots\theta^{(i)}(g)^{\binom{n}{i}}\ldots\theta^{(n)}(g)^{\binom{n}{n}}$$

because if

$$\psi^{(k)}(g) = g^{\binom{k}{0}}\theta(g)^{\binom{k}{1}}\ldots\theta^{(i)}(g)^{\binom{k}{i}}\ldots\theta^{(n)}(g)^{\binom{k}{k}}$$

then

$$\begin{aligned}
\psi^{(k+1)}(g) &= \psi(\psi^{(k)}(g)) = \psi(g^{\binom{k}{0}}\theta(g)^{\binom{k}{1}}\ldots\theta^{(i)}(g)^{\binom{k}{i}}\ldots\theta^{(n)}(g)^{\binom{k}{k}}) \\
&= \psi(g^{\binom{k}{0}})\psi(\theta(g)^{\binom{k}{1}})\ldots\psi(\theta^{(i)}(g)^{\binom{k}{i}})\ldots\psi(\theta^{(k)}(g)^{\binom{k}{k}}) \\
&= (g^{\binom{k}{0}}\theta(g)^{\binom{k}{0}})(\theta(g)^{\binom{k}{1}}\theta^{(2)}(g)^{\binom{k}{1}})\ldots(\theta^{(i)}(g)^{\binom{k}{i}}\theta^{(i+1)}(g)^{\binom{k}{i}})\ldots(\theta^{(k)}(g)^{\binom{k}{k}}\theta^{(k+1)}(g)^{\binom{k}{k}}) \\
&= g^{\binom{k}{0}}\theta(g)^{\binom{k}{0}+\binom{k}{1}}\ldots\theta^{(i)}(g)^{\binom{k}{i-1}+\binom{k}{i}}\ldots\theta^{(k)}(g)^{\binom{k}{k-1}+\binom{k}{k}}\theta^{(k+1)}(g)^{\binom{k}{k}} \\
&= g^{\binom{k+1}{0}}\theta(g)^{\binom{k+1}{1}}\ldots\theta^{(i)}(g)^{\binom{k+1}{i}}\ldots\theta^{(k)}(g)^{\binom{k+1}{k}}\theta^{(k+1)}(g)^{\binom{k+1}{k+1}}
\end{aligned}$$

If p is a prime and $|Z(G)| = p$ then

$$p \mid \binom{p}{1}, \ldots, \binom{p}{p-1}.$$

Hence $\psi^{(p)}(g) = g\theta^{(p)}(g)$, [notice that $\theta(g), \ldots, \theta^{(p-1)}(g) \in Z(G)$]. We want to prove $\psi^{(p(p-1))}(g) = g$ for all $g \in G$.

Let $g \in G$ then we have three state:

i) If $\theta(g) = e$ then $\psi(g) = g$, so $\psi^{(p(p-1))}(g) = g$.

ii) If $\theta^2(g)=e$ then $\psi^2(g)=g\theta(g)^2\theta^{(2)}(g)=g\theta(g)^2$ and by using induction we can prove that $\psi^{(k)}(g)=g\theta(g)^k$ for each k. Let $k=p$, then $\psi^{(p)}(g)=g\theta(g)^p=g$, hence $\psi^{(p(p-1))}(g)=g$.

iii) In this state $\theta(g)\neq e$ and $\theta^{(2)}(g)\neq e$. Let $\theta(g)=a\in Z(G)$ then $\theta^{(2)}(g)=a^i$ for some $1\le i\le p-1$, again we can prove inductively $\theta^{(k)}(g)=a^{i^{k-1}}$, so $\theta^{(p)}(g)=a^{i^{p-1}}=a=\theta(g)$ and $\psi^{(p)}(g)=g\theta^{(p)}(g)=g\theta(g)=\psi(g)$, and again $\psi^{(p-1)}(g)=g$, so $\psi^{(p(p-1))}(g)=g$.

The above statements deduce $\psi^{(p(p-1))}(g)=I$ for every $\psi\in Z(Aut(G))$, therefore

$$Z(Aut(G))^{p(p-1)}=\langle I\rangle$$

as required.

IV. CONCLUSION

In this paper we have proved a theorem on order of $Z(Aut(G))$ when $|Z(G)|=p$. Our results are in fair agreement with other theoretical results reported by other research groups.

REFERENCES

[1] T. W. Hungerford, Algebra, Springer-Verlag,Berlin, 1989.
[2] D. J. S. Robinson, A course in the theory of groups, Springer-Verlag, Berlin, 1982.
[3] J. S. Rose, A course on group theory, Cambridge University Press, 1978.
[4] J. S. Lomont, Applications of Finite Groups, Academic Press, New York, 1959.
[5] Y. G. Smeyers, "Introduction to group theory for non-rigid molecules," Adv. Quantum Chem., vol. 24, pp. 1-77, 1991.
[6] A. R. Ashrafi, M. Hamadanian, "The full non-rigid group theory for tetraaminoplatinum (II)," Croat. Chem. Acta, vol. 76, pp. 299-303, 2003.
[7] A. R. Ashrafi, M. Hamadanian, "Group theory for tetraammine platinum (II) with C_{2v} and C_{4v} point group in the non-rigid system,"
J. Appl. Math. & Computing, vol. 14, pp. 289-303, 2004.