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TRANSCENDENTAL SOLUTION OF FOKKER-PLANCK EQUATION OF VERTICAL GROUND WATER RECHARGE IN UNSATURATED HOMOGENEOUS POROUS MEDIA

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ABSTRACT

The Solution of non linear Fokker-Planck equation of vertical ground water recharge in unsaturated Porous Media is obtained in terms of transcendental function. The mathematical formulation leads to nonlinear differential equation which has been reduced to well –known Abel's equation of the second kind by using a similarity transformation. A formal mathematical solution of the later has been obtained in terms of transcendental functions. The solution represents moisture content in vertical downward direction at any depth z for any time t > 0.

Key words: Transcendental, Abel's equation, Moisture content, Fokker-Planck equation

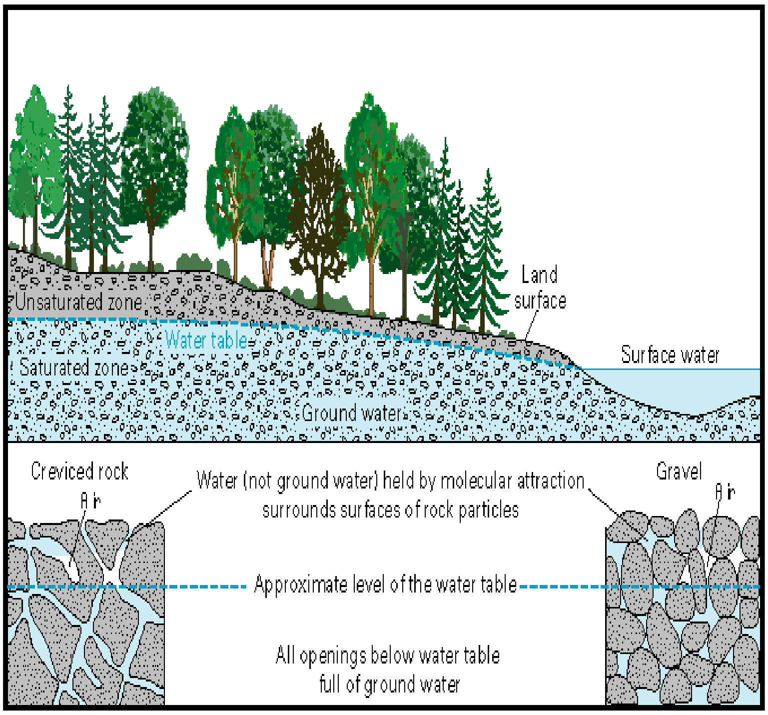
I. INTRODUCTION

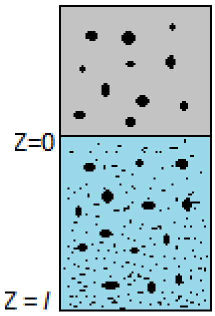
In this paper flow of water in unsaturated soil has been considered with some simplifying assumptions. In typical soil profiles some distance separates the earth's surface from the water table, which is the upper limit of completely water-saturated soil. In this downward direction the water saturation varies between 0 and 1. Water flow in this *unsaturated zone* is complicated by the fact that the soil water diffusivity depends on its saturation (Allen (1985)).

The flow of water through soils in many practical situations is unsteady and slightly saturated. It is unsteady because the moisture content changes as a function of time. It is slightly saturated because all the pore spaces are not completely filled with flowing liquid. Example of such flows are the infiltration of water through the ground surface, the flow through the capillary fringe of an unconfined aquifer, draining soils, evaporation from an aquifer close to the ground surface, ground water level fluctuations, the inflow of water from irrigation channels and the underground disposal of sewage and waste. Since it is important to know the water content of the soils in these flows, solutions to the equation describing such flows are very useful to many branches of engineering, such as civil, hydrologic, sanitary and irrigation. The equation which describes the above multi-dimensional flows is a non-linear partial differential equation.

The phenomenon of the one dimensional vertical ground water recharged by spreading is of great importance for hydrologists, agriculturists and for people related with water resources sciences. This phenomenon has been discussed by many researchers taking different viewpoints. Here are some of the examples-Klute (1952) employed a finite difference approach by taking a linearization to reduce diffusion equation to an ordinary differential equation and has employed a forward integration and iteration method. Other researchers have examined diffusion equation with a linear diffusion coefficient. Swartzendruber used Philip's (1970) method to get graphical illustration of the mathematical solution for horizontal water function. Verma(1969) has solved by Laplace transformation technique; Mehta (1975) obtains an approximated solution by the method of singular perturbation technique. He considered the average diffusivity coefficient of the whole range of moisture content and treated as small constant. Allen(1985) had reviewed modification, assumptions and different techniques used by researchers. Hari Prasad *et al.* (2001)

had developed a numerical model to simulate water flow through unsaturated zones and study the effect of unsaturated soil parameters on water movement during different processes such as gravity drainage and infiltration to the variations in the unsaturated soil parameters. This model is also applied to predict moisture contents during a field internal drainage test. De Vries and Simmers (2002) has discussed processes and challenges principally on recharge of unconfined aquifers, often the most readily available and affordable source of water in (semi-) arid regions. Faybishenko (2004) has given review of the theoretical concepts, has presented the results, and provided perspectives on investigation of flow and transport in unsaturated heterogeneous soils and fractured rock, using the methods of nonlinear dynamics and deterministic chaos. Patel and Mehta (2006) studied the ground water recharge by spreading in vertical downward direction. They constitute governing differential equation as Burger's equation, with permeability as nonlinear function of moisture content.





II. STATEMENT OF PROBLEM

In the investigated model it is considered that the ground water recharge takes place over the large basin taken as homogeneous porous medium. Consider the water of rainfall entered in downward direction through unsaturated porous media and infiltration takes place at z=0 which is common interface between saturated and unsaturated zone as shown in figure [1]. In this case the flow may be assumed vertically downward neglecting spreading in other directions (i.e. small amount of water may spread in other directions but it is very small compared to large size of basin) through slightly saturated porous media i.e. initial saturation is non-zero. The rate of change permeability of the medium with respect to depth z is considered to vary directly with rate of change in moisture content and ratio of depth and time. In the present analysis the soil water diffusivity $D(\theta) = \frac{a}{(b-\theta)^2}$. (Broadbridge & White [1984])

The following assumptions have been made for present analysis (Hari Prasad *et al.*(2001):

The medium is homogenous. There is no air resistance to the flow (i.e. the porous medium contains only the flowing liquid water and empty voids). The air in the void space is approximately at atmospheric pressure i.e. *air is stationary*. The soil properties are taken to be constant. The flowing liquid (water) is considered continuous at a microscopic level, incompressible and isothermal. The initial moisture content is uniform throughout the soil profile and the moisture content at the soil surface is constant and near saturation or rainfall or irrigation rate is constant. Darcy's law is applicable.

The governing equation which describes the one-dimensional flow with above mentioned assumptions is a non-linear partial differential equation which has been reduced to Abel's equation of the second kind by using a similarity transformation and then solved in terms of transcendental function.

III. MATHEMATICAL FORMULATIONS

The motion of water in isotropic, homogeneous porous medium is given by Darcy's law as, (Bear 1972)

$$\vec{v} = -k\nabla H \tag{3.1}$$

Where , $\vec{v} = Volume\ flux\ of\ moisture$

k = coefficient of soil water diffusivity

$\nabla H = gradient\ of\ whole\ moisture\ potential.$

$$\frac{\partial}{\partial t}(\rho_s PS) = -\nabla M \tag{3.2}$$

Where ρ_s is the bulk density of the medium, P is the porosity, S is saturation of water and M is the mass of flux of water at any time $t \geq 0$.

Using the incompressibility of water and considering the fact that the moisture content $\theta = PS$ (John 1976), equation (3.2) reduces to

$$\frac{\partial}{\partial t}(\rho_s \theta) = -\nabla(\rho \vec{v}) \tag{3.3}$$

Where ρ is the flux density.

Substituting value of $\vec{v}$ from equation (3.1) in to equation (3.3), we get

$$\frac{\partial}{\partial t}(\rho_s \theta) = \nabla(\rho k \nabla H) \tag{3.4}$$

As in the present problem, flow is taken in vertical downward direction (Swaroop 2002) equation (3.4) can be written as

$$\rho_s \frac{\partial \theta}{\partial t} = \rho \frac{\partial}{\partial z}\left(k \frac{\partial H}{\partial z}\right) \tag{3.5}$$

Further considering the relation $H = \psi + gz$ (John 1976), where ψ is the capillary pressure potential,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\rho}{\rho_s} k \frac{\partial \psi}{\partial z} + \frac{gk\rho}{\rho_s}\right) \tag{3.6}$$

It is well known that capillary pressure potential and moisture content are related by single valued function (John 1976). The equation (3.6) results in to

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) + \frac{g\rho}{\rho_s}\frac{\partial k}{\partial z} \tag{3.7}$$

Where $D(\theta)$ is the soil water diffusivity coefficient given by $D(\theta) = \frac{\rho}{\rho_s} k \frac{\partial \psi}{\partial z}$

Equation (3.7) can be written as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] + \frac{g\rho}{\rho_s} K'(\theta)\frac{\partial \theta}{\partial z} \tag{3.8}$$

Which is known as non-linear Fokker-Planck diffusion-convection equation (Broadbridge & White [1984])

Where t is time, z is the depth (positively downward), $\theta(z,t)$ is volumetric soil water content, $D(\theta)$ is the soil water diffusivity, $K(\theta)$ is the hydraulic conductivity and

$$K'(\theta)=\frac{dK}{d\theta}$$

This equation (3.8) is model based on Darcy-Buckingham approach to unsaturated water flow in unsaturated zone as fig [1].

It assumes simple functional forms for the soil water diffusivity $D(\theta)$ and hydraulic conductivity $K(\theta)$, which are given by the relation

$$D(\theta)=\frac{a}{(b-\theta)^2} \quad , \qquad K(\theta)=\beta+\gamma(b-\theta)+\frac{\delta}{2(b-\theta)} \qquad \text{(Broadbridge \& } White \text{ [1984])}$$

$$D(\theta)=\frac{a}{b^2}\left(1-\frac{\theta}{b}\right)^{-2}=\frac{a}{b^2}\left(1+\frac{2\theta}{b}\right) \quad since\ \theta\ is\ very\ small. \tag{3.9}$$

and

$$K(\theta)=\beta+\gamma b+\frac{\delta}{2b}+\left(\frac{\delta}{2b^2}-\gamma\right)\theta \tag{3.10}$$

Where β, γ, λ, a, b, are constant.

Using equation (3.9) & (3.10) in (3.8), we get

$$\frac{\partial\theta}{\partial t}=\frac{a}{b^3}\frac{\partial}{\partial z}\left((b+2\theta)\frac{\partial\theta}{\partial z}\right)+\frac{\rho g}{\rho_s}\left(\frac{\delta}{2b^2}-\gamma\right)\frac{\partial\theta}{\partial z} \tag{3.11}$$

This describes the non-linear Fokker- Planck diffusion-convection model describing constant rate rainfall infiltration in uniform soil and other porous material.

The appropriate boundary conditions are,

$$\theta(0,t)=\theta_0\ ;\ for\ any\ t>0 \tag{3.12}$$

$$\theta(z,0)=\theta_1\ ;\ for\ 0<z<L \tag{3.13}$$

$$\& \quad \frac{\partial\theta}{\partial z}(0,t)=\varepsilon\ ;-1<\varepsilon<0 \tag{3.14}$$

The condition (3.12) is boundary volumetric soil water at top of the surface. The condition (3.13) is initial volumetric soil water content and (3.14) presents variation in volumetric soil water content at the top of the surface.

For the sake of convenience, we consider that the rainfall infiltration in uniform soil take place in positively downward direction from z = 0 top of the surface to the bottom $z=L$ where water table is saturated as fig. [1].

IV. SOLUTION OF THE PROBLEM

Let $b+2\theta=h(\theta) \implies 2\frac{\partial\theta}{\partial t}=\frac{\partial h}{\partial t}$

Substituting in (3.11),

$$\frac{\partial h}{\partial t} = \frac{a}{2b^3}\frac{\partial}{\partial z}\left(h\frac{\partial h}{\partial z}\right) + \frac{\rho g}{\rho_s}\left(\frac{\delta}{2b^2} - \gamma\right)\frac{\partial h}{\partial z} \tag{4.1}$$

$$\frac{\partial h}{\partial t} = A\frac{\partial}{\partial z}\left(h\frac{\partial h}{\partial z}\right) + B\frac{\partial h}{\partial z} \quad , \quad where\ A = \frac{a}{2b^3} \quad and\ B = \frac{\rho g}{2\rho_s}\left(\frac{\delta}{2b^2} - \gamma\right) \tag{4.2}$$

Their model describes the dependence of the hydraulic conductivity on water content. The analytic solutions describe the temporal development of the water content profile during rainfall. They predict the time dependence of both surface moisture content and surface soil water potential and the shape of large-time.

Equation (4.2) can be written as

$$\frac{\partial h}{\partial t} = A\left[h\frac{\partial^2 h}{\partial z^2} + \left(\frac{\partial h}{\partial z}\right)^2\right] + B\frac{\partial h}{\partial z} \tag{4.3}$$

The boundary conditions (3.12) to (3.14) converted into

$h(0,t) = b + 2\theta_0$, when z = 0 , any t > 0 (4.4)

$h(z,0) = b + 2\theta_1$, when t = 0 , any z (4.5)

$\frac{\partial h}{\partial z}(0,t) = b + 2\varepsilon$, when z = 0 , any t > 0 (4.6)

$$Choose\ \frac{h}{b+2\theta_1} = t\,F(\eta) \quad where \quad \eta = zt^{-1}, \tag{4.7}$$

Combining equation (4.3) and (4.7), we get

$$\left[F(\eta)F''(\eta) + \left(F'(\eta)\right)^2\right] + \left(\frac{\eta + B}{A}\right)F'(\eta) - \frac{1}{A}F(\eta) = 0 \tag{4.8}$$

The boundary conditions (4.4) to (4.6) can be converted in terms of new variables.

$$Taking\ \frac{\eta + B}{A} = \xi \Rightarrow \frac{d\xi}{d\eta} = \frac{1}{A} \tag{4.9}$$

$$F(\xi)F''(\xi) + \left(F'(\xi)\right)^2 + \xi AF'(\xi) - AF(\xi) = 0 \tag{4.10}$$

This is nonlinear ordinary differential equation of second order.

Equation (4.10) can be solved by considering the substitution

$$F(\xi) = \xi^2 u(z); \ z = \log\xi \quad \text{and} \quad u'(z) = p \tag{4.11}$$

then eqn.(4.10) takes the form

$$upp'(u) + p^2 + (7u + A)p + u(6u+1) = 0. \tag{4.12}$$

This is the Abel's equation of second kind whose solution can be obtained by considering the substitution.

$$up = \log v(z) \tag{4.13}$$

Substituting eqn. (4.13) into eqn. (4.12), we get

$$\frac{1}{v}\frac{dv}{du} + (7u + A)p + (6u + 1)u = 0. \tag{4.14}$$

Further supposing that log v(z) = M(z), the eqn. (4.14) can take the form

$$\frac{dM}{dz} + (7u + A)\frac{M}{u} + u(6u + 1) = 0 \tag{4.15}$$

The corresponding boundary conditions can be converted in terms of new variables.

The solution of eqn. (4.15) is

$$Me^{\int\left(\frac{7}{\xi}+\frac{A\xi}{F}\right)d\xi} = \int \frac{F}{\xi^3}\left(1+\frac{6F}{\xi^2}\right)e^{\int\left(\frac{7}{\xi}+\frac{A\xi}{F}\right)d\xi} d\xi + C \tag{4.16}$$

where C is a constant of integration.

For evaluating the constant C, we can use the following boundary condition:

At $t = 0$, $\xi = \infty$ and $F(\xi) = \infty$

Therefore, we get $C=0$.

The eqn. (4.15) takes the form

$$Me^{\int\left(\frac{7}{\xi}+\frac{A\xi}{F}\right)d\xi} = \int \frac{F}{\xi^3}\left(1+\frac{6F}{\xi^2}\right)e^{\int\left(\frac{7}{\xi}+\frac{A\xi}{F}\right)d\xi} d\xi \tag{4.17}$$

Equation (4.16) gives the formal solution in terms of transcendental functions.

V. CONCLUSIONS

The solution of equation (4.15) is equation (4.17) which gives formal solution in terms of transcendental function. Substituting independent and dependent variable in equation (4.17), the analytical solution is obtained for non-linear Fokker-Planck diffusion-convection model which represents the volumetric soil water content for different depth z for any time t > 0.In this paper we have obtained the formal solution of vertical ground water recharge in unsaturated porous media by using similarity transformation. The mathematical formulation leads to a non linear differential equation which has been reduced to Abel's equation of second kind and finally its analytical solution is obtained in terms of transcendental functions. We have not included any numerical illustration or graphical representation due to our particular interest but the same can be done by using the field values of the characteristic parameters.

VI. NOMENCLATURE

$\vec{v} = Volume\ flux\ of\ moisture$

ρ_s = bulk density of the medium

ψ = capillary pressure potential

$K(\theta)$= hydraulic conductivity

$\nabla H = gradient\ of\ whole\ moisture\ potential$

k = coefficient of soil water diffusivity

M = mass of flux of water

P = porosity

S = Saturation of water

t= time

$D(\theta)$ = soil water diffusivity coefficient

$\theta =$ moisture content

ρ = the flux density

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REFERENCES

Allen, M.B.(1985). Numerical modelling of multiphase flow in porous media. *Adv. Water Resources*, Volume 8, December, 162-187.

Bear, J.(1972). *Dynamics of Fluids in Porous Media*. American Elsevier Publishing Company, Inc.

C. Rogers, M. P. Stallybrass, D. L. Clements (1983): On two phase filtration under gravity and with boundary infiltration: Application of a Backlund transformation, Nonlinear Anal. Theory methods Appl,.7,pp. 785-799.

De Vries, J.J.and Simmers, I.(2002). Groundwater recharge: an overview of processes and challenges. *Hydrogeology Journal.* 10,5-17.

Faybishenko, B.(2004). Nonlinear dynamics in flow through unsaturated fractured porous media: Status and perspectives. *Rev. Geophys.* 42, RG2003, doi:10.1029/2003RG000125.

Hansen, A.G.(1964) *Similarity analysis of boundary value problem in engineering.* Prentice Hall of Canada, Ltd..

Hari Prasad, K.S., Mohan kumar, M.S. and Shekhar, M(2001). Modelling Flow Through Unsaturated Zone: Sensitivity to Unsaturated Soil Properties. *Sadhana, Indian Academy of Sciences,* 26(6), 517-528.

John, B.C.(1976). Two dimensional unsteady flow in unsaturated porous media, *Proc. 1st Int. Conf. Finite Elements in Water Resources,* Princeton University, U.S.A. pp3.3-3.19.

Klute, A.A.(1952). Anumerical method for solving the flow equation for water in unsaturated materials. *Soil Science,* pp 73-105.

M. Awad. I. Turner (2000): Flux-limiting and non-linear solution techniques for simulation of transport in porous media, ANZIAM J. 42,pp. 157-182.

Mehta M. N. (1975): Multiple scales solution of one dimensional flow in unsaturated porous media, Indian journal of Engg. Maths, Vol. 5, 1.

Mehta, M.N.(1975). A single perturbation solution of one-dimensional flow in unsaturated porous media. *Proc. FMFP,* $E_1 - E_2$.

Mehta, M.N.and Yadav Saroj R.(2007). Solution of problem arising during vertical groundwater recharge by spreading in slightly saturated porous media. *Journal of Ultra Scientist of Physical Sciences,* Vol.19(3) M, 541-546.

P. Broadbridge, I. White, (1984): Constant rate rainfall infiltration: A Versatile nonlinear model 1, Analytic solution, Water Resour. Res. 24, pp. 145-154.

Perkins, T.K. and Johnston, O.C.(1963). A review of diffusion and dispersion in porous media, *Soc. Pet. Engng J.*, 3,70-80.

Philips, R.(1970). *Advance in Hydro-science* (edited Ven Te Chow). Acad. Press, New York, 6.

Pinder, G.F. and Cooper, H.H.Jr.(1970). A numerical technique for calculating the transient position of the saltwater front, *Water Resources Research,* 6,875-882.

Polubainova-Kochina, P. Ya. (1962). *Theory of Groundwater movement.* Princeton University PRess, pp 499-500.

Swaroop, A.A.(2002). *Numerical Simulation of Problem Arising in Fluid Flow through Porous Media.* Ph.D. Thesis, South Gujarat University, Surat, India, 125-138.

Verma, A.P.(1973). A similarity solution of a unidimensional vertical ground water recharge through porous media. *Rev.Roum.Scien.Techn.Ser.Mec.Appl.* 18, 2,345-351.